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A NEW ANALYSIS OF

PLANE GEOMETRY

FINITE AND DIFFERENTIAL

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A NEW ANALYSIS OF

PLANE GEOMETRY

FINITE AND DIFFERENTIAL

WITH NUMEROUS EXAMPLES

BY

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PREFACE

IT is the purpose of this work to present a new analysis of Plane Geometry. We know that any geometrical theorem may be expressed as a relation in points. We may however look upon Plane Co-ordinate Geometry as having points and lines for its fundamental elements; in relations of which geometrical theorems are going to be expressed. Thus the equation $y = mx + c$ may be looked upon as a line of co-ordinates (m, c). It is this view that we shall adopt. Now let us denote points by small Latin letters and lines by small Greek letters. Let $a, b, c \ldots l$ be a set of points; $\alpha, \beta, \gamma \ldots \lambda$ a set of lines. Let us denote the joins and intersections of two points and two lines respectively by drawing a bar over them, thus $\overline{ab}$, $\overline{\alpha\beta}$. Also let us denote the distance of two points a, b by (ab); the perpendicular distance from a on β by $(a\beta)$; the angle between α, β by $(\alpha\beta)$. Let us use the term 'measure' to include the three cases. Let us use the notation (x_a, y_a) to denote the co-ordinates of a and $(\xi_\alpha, \eta_\alpha)$ to denote the co-ordinates of α.

Then what Co-ordinate Geometry effects is the reduction of expressions such as

$$(\overline{\overline{ab}\gamma}\delta \ldots \overline{\overline{\overline{ef}\lambda}h}\mu \ldots) \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}$$

to a function of (x_a, y_a), $(x_b, y_b) \ldots (\xi_\alpha, \eta_\alpha) \ldots$.

Now let ρ, σ be the Cartesian axes, then

$$x_a = (a\rho),\ \ y_a = (a\sigma),$$

and we may put

$$\xi_\alpha = (\alpha\rho),\ \ \eta_\alpha = (\overline{\rho\sigma}\alpha),$$

i.e. the perpendicular from the origin on α.

Hence such an expression as (A) is reduced to a function of measures of the elements occurring taken singly with the reference axes.

This idea I have generalized and have reduced the expression to measures of the elements taken in pairs. On p. 109 is an important result which I regret does not appear with Chapter I (it should be read between §§ 20, 21), which states that the square of distances, that perpendicular distances and the sine and cosine of angles are reducible to the quotient of two polynomials in

(1) moduli of measures of two points, Ex. $|(ab)|$,

(2) measures of a point and line, Ex. $(a\beta)$,

(3) sines of measures of two lines, Ex. $\sin(\alpha\beta)$,

(4) cosines of measures of two lines, Ex. $\cos(\alpha\beta)$,

(5) measures of the join of two points and a point, Ex. $(\overline{ab}c)$,

(6) measures of the intersection of two lines and a line, Ex. $(\overline{\alpha\beta}\gamma)$.

(7) cosines of the join of two points and a line, Ex. $\cos(\overline{ab}\gamma)$.

Cases (5) and (7) are reduced in surd form and (6) by means of a point to measures of two elements. Thus we do away with the idea of reference elements.

But this brings us to another matter. We know that taking four arbitrary points, there is a relation between the six pairs of measures of two elements. We have also such relations in the case of three points and a line, two points and two lines and in the case of three lines. We have called such relations eliminants, being eliminants of relative position. Suppose we have reduced all our complex measures and noted all our eliminants. The matter of proving a relation between the complex measures reduces to proving the relation between their reductions with the help of the eliminants.

Again, a point may be got from another point by a vectorial construction. We have denoted by $a_{\omega,\rho}$ the point distant ρ from a and measured in the direction of ω. We include such derived points in our consideration. To do this formally we take the point of general form $a_{\omega_1,\rho_1,\omega_2,\rho_2\,.\,\omega_n,\rho_n}$ denoting the point derived by a succession of such constructions. For the corresponding

line we take $a_{\omega_1,\rho_1;\,\omega_2,\rho_2\,.\,\omega_n,\rho_n;\,\omega}$ denoting the line parallel to ω and passing through $a_{\omega_1,\rho_1;\,\omega_2,\rho_2\,..\,\omega_n,\rho_n}$.

Now take any complex measure containing such vectorially derived elements. Thus

$$(\overline{\overline{a b_{\omega_1,\rho_1,\,\omega_2,\rho_2}\gamma}\,\ldots\,k_{\lambda_1,\sigma_1,\,\lambda}}\,\overline{\overline{\alpha c_\omega} d}\ldots).$$

This as we have seen is reducible to a function of measures of pairs of fundamental elements and vectorially derived elements.

In Chapter II we have reduced

$$(a'b)^2,\ (a'\beta),\ \text{where}\ a' = a_{\omega_1,\rho_1,\,\omega_2,\rho_2\,.\,\omega_n,\rho_n},$$

$$(\alpha'b),\ (\alpha'\beta),\ \text{where}\ \alpha' = a_{\omega_1,\rho_1,\,\omega_2,\rho_2\ \ \omega_n,\rho_n,\,\omega},$$

to functions of measures of pairs of fundamental elements and $\rho_1, \rho_2 \ldots \rho_n$; including $\omega_1, \omega_2 \ldots \omega_n$ in the fundamental elements.

Hence we can reduce any complex measure as above to a function of measures of pairs of fundamental elements and the magnitudes occurring in the vectors.

We shall consider yet another class of derived elements. These are elements derived from the fundamental elements by an equation. Thus from the lines $\alpha_1, \alpha_2 \ldots \alpha_n$ we have the derived line $\Sigma a_r (x\alpha_r) + a = 0$.

Again from the points $a_1, a_2 \ldots a_n$ and lines $\beta_1, \beta_2 \ldots \beta_m$ we have the derived point

$$\Sigma A_r (\xi a_r) + \Sigma B_r \cos(\xi\beta_r) = 0,$$

where $A_1, A_2 \ldots A_n, B_1, B_2 \ldots B_m$ are algebraic magnitudes.

In Chapter III it is shewn that these can be treated in a similar way to that indicated for vectorially derived elements.

Thus if our Geometry comprise only elements derived from fundamental elements by (i) intersections and joins, (ii) by vectorial constructions, (iii) by equational relations, we can reduce any measure of such elements to a function of measures of two elements and algebraic magnitudes. We note the eliminants. We may also have imposed relations stated in the particular problem. With these conditions we must prove relations between certain measures. This is a complete statement of our problem. We have thus stated our problem as a matter of reductional computation.

The method devised for Differential Geometry is of a similar character.

Let x be a moving point and let x' be its consecutive position. This displacement of x we shall define by its direction of displacement and the amount of its displacement: or in our notation $\overline{xx'}$ and $|(xx')|$. We denote $\overline{xx'}$ by the operative notation τx and $|(xx')|$ by δx.

Similarly let ξ be a moving line and ξ' a consecutive position. The point of intersection $\overline{\xi\xi'}$ we shall denote by the operative notation $p\xi$ and the amount of angular displacement $(\xi\xi')$ by $\delta\xi$.

We shall first consider Differential Geometry of one displacement. Any problem dealing with such Geometry may be reduced to the consideration of such a measure as

$$(\overline{\overline{p\bar{a}bc}\,\tau\Sigma a_r\,(x\alpha_r) = 0}\ldots\overline{\bar{p}x_{\rho\sigma}\quad\overline{{}_{\phi\omega}k}\,\tau x_{\phi}\ldots}),$$

and the differential of any complex measure or element.

The method of reducing such an expression is developed in Chapters IV—VIII. By means of the principle thus set forth we are enabled to reduce such a measure to measures of the fundamental elements and elements of displacement τa, $\tau b\ldots$, $p\alpha$, $p\beta\ldots$ and the amount of their displacement. We may look upon τa, τb . ., $p\alpha$, $p\beta\ldots$ as fundamental elements. We take note of the eliminants of all our fundamental elements and these elements of displacement. The magnitude of the displacement of each element we must look upon simply as small algebraic quantities. Should there be any imposed conditions we have these and also their first derivatives. Our problem is then completely stated and set forth as a matter of reductional computation.

We next come to the matter of elements of displacement of elements of displacement, such as $p\nu x$ when νx is the line through x perpendicular to τx. We suppose all the fundamental elements that are variable to trace continuous curves and in Chapter VIII we have shewn how to reduce such to elements and differentials of first displacements and the curvature of the curve at the elements.

We next consider elements of displacement of elements of displacement of elements of displacement and so on. All

quantities which depend only on the curve traced by a variable point we shall call intrinsic functions of the point. In Chapter VIII it is shewn that such are reducible to elements of a single displacement and intrinsic functions of the curve.

Thus any such measure as

$$(\overline{p\nu^{m}\overline{\overline{\overline{ab}\gamma}d} \dots \nu^{n}\Sigma\, a_r\,(x\alpha_r) = 0 \dots x_{\phi\theta}})$$

is reducible to measures of fundamental elements and elements of displacement of these elements and intrinsic functions of these elements. We also need the differentials of such measures and such derived elements as

$$\overline{p\nu^{m}\,\overline{\overline{\overline{ab}\gamma}d} \dots \nu^{n}\Sigma\, a_r\,(x\alpha_r) = 0}.$$

These problems are the most general problems of Differential Geometry. The last chapter is a chapter on Integration adopting these ideas.

In the Miscellaneous Examples I have endeavoured to illustrate the method. As the present work is intended as a presentation of method, I have not tried to make the examples exhaustive of well-known properties. The large modern theory of singularities of curves I have not considered at all.

I must apologise for the rather amateurish manner of the statement of the axioms, which are ticketed with large Roman numerals. These are the axioms which form the basis of the symbolic procedure of the text. This method must not be worked out by using a figure. The result is generally a hopeless quandary of sign. It must be worked directly from the axioms and deductions after having translated the conditions of the figure into a statement in symbols. As will be seen, no more than these axioms are required so far as the domain defined is concerned. The axioms may be divided into two main classes from a natural standpoint, axioms of actual properties and axioms of convention. The latter have from time to time been given by successive writers for the purpose of comprehending many cases in one though I believe they appear in a connected form for the first time here. There are two main points in regard to a system of axioms. The first is that they should be sufficient, the second

that they should be consistent. The proof that they are consistent I have not attempted. The fact that they have not yet yielded a contradiction is a powerful argument of their consistence.

Axioms (XV), (XVIII) have been proved visually. But no doubt with a few more fundamental axioms of superposition and orientation, these could be deduced.

There are two main ways of considering Geometry. One is by sight or figure. The other is by a symbolic representation of the figure. The former method I shall call the Visual method: the latter the Symbolic method.

Visual Geometry, as it is known, is of a *synthetic* or *transformational* character but there is no reason why it should not be *analytic* or *reductional.* A few cases can be cited in which a theorem in Visual Geometry is worked out by a reductional process. The conditions of the problem may be represented by certain magnitudes determined by the problems, such as areas, distances: and what is to be proved is also a relation between the *same* magnitudes. The working after this is a matter of reductional computation.

As an alternative method to the Visual method we have the Symbolic method. As a rule the method of Symbolic Geometry is of a reductional character.

The method of the text belongs to the Symbolic method. In some cases, however, the process is easily visualised and coincides with the treatment of Visual Geometry.

As regards its accomplishments, the method of Visual Geometry manifests unexpected power, a power which, however, is not sustained. An illustration of this power is afforded by Hart's proof of Steiner's construction of Malfatti's problem.

The Symbolic method is characterized by its complete grasp of the problem: compared with the method of Visual Geometry it lacks the power of its transformations.

The method of the text has an advantage over Co-ordinate Geometry in the matter of sign. The method of the text gives a more automatic account of sign than does Co-ordinate Geometry,

as the text will I think shew. In a Cartesian system, a line is represented by an equation. Now an equation gives no 'sense' to a line which I think explains this deficiency. Casey has given a convention which applies to Co-ordinate Geometry which states that the perpendicular from a point on a line is positive or negative according as it is on the same or opposite side of the origin, but it does not seem to have been developed in conjunction with others.

I have been engaged on the present work for the last three years. I claim the method as original. There are some theorems which are original, and most of the general results in the examples I believe are new. The new treatment of the trigonometric functions is original and necessary for the purpose.

We notice here the almost identity in symbol between the method of Grassmann and that of the text, in the case of Geometry of Position. The theorems and proofs on cubic curves in the *Miscellaneous Examples* have been adapted directly from Grassmann's theorems and proofs as given in Whitehead's *Universal Algebra*. I was not aware of the method of Grassmann before I discovered the method of the text, though I was aware of a similar method which the *Algebra of Invariants* affords.

Not many cases of a general transformation have occurred. One of the best is Example 12, § 28. Most of the Examples given on *parallique* and *orthologique* pairs of triangles are readily proved from this.

I have laboured to eliminate errors of detail, but no doubt in a new work like this there are some still remaining.

The notation I hope will meet with approval. My aim has been to make it unambiguous, easily written and as short as possible. The notation of putting τ before x for the direction of displacement of x is ambiguous unless we agree to reserve τ for this special purpose.

In the Appendix I have given four cases of reduction of products of measures. Each is such that, though a component measure is reducible only by radicals, owing to the eliminants the product can be expressed without radicals.

It is a great pleasure to me to acknowledge my obligations to Mr S. Chapman, Fellow of Trinity College, Cambridge, for reading part of the proofs with me and for suggestions. The terms 'measure' and 'determinate' are due to him.

In conclusion I wish to express my gratitude to the readers and officials of the University Press for their close attention and unfailing courtesy.

A. W. H. THOMPSON.

April, 1914.

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FINITE GEOMETRY

INTRODUCTION

§ 1. *Definitions:* (1) The elementary concepts of plane Geometry are points and lines. We shall refer to them as *elements.*

(2) That a line passes through a point we shall also express by saying that the line is *incident** in the point, or the point is *incident* in the line.

(3) The *determinate* of two elements of like kind is the element which is incident in both these elements. Thus the determinate of two points is the line joining them, and the determinate of two lines is their point of intersection.

(4) The *measure* of two elements is a certain quantity determined by these elements, expressing the relation of one in regard to the other. The measure of two points is the distance between them. The measure of a point and a line is the perpendicular distance between the point and line. The measure of two lines is the angle between them.

The question of the sign of measures is fundamental.

§ 2. *Notation.* Points will be denoted exclusively by small Latin letters $a, b, c \ldots x, y, z$: lines by small Greek letters $\alpha, \beta, \gamma \ldots \omega$.

The determinate of two elements will be denoted by writing them side by side and drawing a bar over the two. Thus the determinate of a point a, and a point b will be denoted by $\overline{ab}$. The determinate of a line α and a line β by $\overline{\alpha\beta}$.

The measure of two elements will be denoted by writing them side by side and enclosing them in small brackets. Thus the measure of a point a and a line b is written (ab). The measure of a point a and a line β is written $(a\beta)$, and of two lines $(\alpha\beta)$.

* See Whitehead's *Axioms of Geometry.*

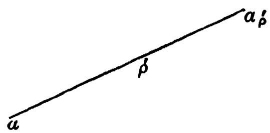
Vector will be used in the ordinary sense. A vector will be denoted by a small Greek letter with an acute accent. Thus $\acute{\rho}$ will denote a vector. The direction of the vector will be denoted by ρ and the magnitude by ρ with a circumflex accent, thus $\hat{\rho}$.

The letters μ, ν, τ will be used for certain special purposes.

§ 3. A point may be derived from given elements in the following three ways:

(i) by a scheme of determinates only. This is the way points are obtained from other elements in Descriptive Geometry. Thus $\overline{\overline{ab}\gamma}$ gives a new point, namely the intersection of the line $\overline{ab}$ with the line γ.

(ii) by a scheme of vectors. Thus if $\acute{\rho}$ denote a vector and a a point, we get another point $a_{\acute{\rho}}$. This denotes the point distant $\hat{\rho}$ from a, measured in the direction of ρ.



(iii) by an equation, as is done in co-ordinate Geometry. Thus if a_1, $a_2 \ldots a_n$ be n points, and ξ a variable line, and $A_1, A_2 \ldots A_n$ algebraic magnitudes, the equation $\Sigma A_r (\xi a_r) = 0$ denotes a point; meaning that any line which satisfies the equation passes through a fixed point.

There are other ways of getting new points, as by the rolling of one curve upon another; but the above three are the only ones we shall consider.

Again, a new line may be derived from a set of points and lines in three corresponding ways:

(i) by a scheme of determinates.

(ii) by a scheme of vectors and direction. Thus if a is a point and ρ a direction, a_ρ is the straight line through a, with direction ρ.

(iii) by an equation. If $\alpha_1, \alpha_2 \ldots \alpha_n$ be n lines and x a variable point, and $a_1, a_2 \ldots a_n$ algebraic quantities, the equation $\Sigma a_r (x\alpha_r) = 0$ denotes a line.

§ 4. In the present theory a geometrical property depends on an equation in measures. Now the proof that one measure equals one or more other measures, may be effected by reducing

all the measures to measures of two elements. Also we have, with one exception, for any four unrelated elements, a relation between the measures of the six possible pairs. The proof after this is a matter of algebra. However it is not necessary always to *reduce* the measures. It would suffice if we could so *transform* the measures, without actual reduction, so as to shew their equality. The former method of *reduction* corresponds to Analytical Geometry; the latter method of *transformation* corresponds to Synthetic Geometry.

In the text the method of *reduction* is used uniformly. Our object will be then to classify measures and give a calculus for their reduction. It will be seen that the formulae of Co-ordinate Geometry depend for their use on the fact that they enable one to reduce measures Instead of these formulae, we have given the actual reduction of such measures as are necessary, and give a definite method for the reduction of more complex ones from these.

§ 5. *Sketch of Method.*

The measures of two elements are fundamental. They are three in number.

The measures containing three elements, we shall call measures of the third order. These are, with one exception, reducible, that is to say, they can be reduced to algebraic or trigonometric functions of the measures of two elements.

The measures of two elements are (xy), $(x\eta)$, $(\xi\eta)$ where x, y are points and ξ, η lines.

The order of a function of several measures we define as equal to the greatest number of elements occurring in any component measure of the function: thus $|(xy)|\,(\overline{xy}z)$ is a measure of the third order; here $|(xy)|$ denotes as usual the modulus of (xy).

It is to be remarked that the order of a measure so defined only applies in the case where no two of the elements are identical.

A measure of three lines is not reducible. However with the introduction of an arbitrary point, it can be reduced.

All measures of the fourth order are reducible. Hence all measures containing three or more elements are reducible.

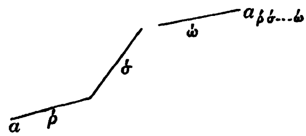
So far we have only dealt with simple points and lines. We have next to consider the two other classes of elements stated in § 3. These we shall call

(i) the general vectorial point and line.

(ii) the general equational point and line.

We shall consider first the general vectorial elements.

The general vectorial point is $a_{\acute\rho\acute\sigma\ldots\acute\omega}$. This denotes the point derived from a by a series of vectors $\acute\rho$, $\acute\sigma$ ... $\acute\omega$, as figured.



The general vectorial line is denoted by $a_{\acute\rho\acute\sigma.\ \acute\phi\omega}$ where $a_{\acute\rho\acute\sigma..\ \acute\phi\omega}$ means the line parallel to ω and passing through the point $a_{\acute\rho\acute\sigma\ldots\acute\phi}$.

As regards the reduction of measures of these vectorial elements, we need only find the values of the measures of two elements, which are

$$(a_{\acute\rho\acute\sigma}\ {}_{\acute\phi}b)^2,\ (a_{\acute\rho\acute\sigma}\ {}_{\acute\phi}\beta),$$
$$(a_{\acute\rho\acute\sigma..}\ {}_{\acute\phi\omega}b),\ (a_{\acute\rho\acute\sigma\ldots\acute\phi\omega}\beta).$$

Having found the reduction of these we can, by the formulae for the reduction of measures of simple elements, reduce the measures of simple and vectorial elements.

We proceed in an exactly similar manner in regard to the reduction of measures of equational elements. In other words we have a calculus for the reduction of measures.

We have so far considered the geometry of finite concepts only.

§ 6. We next indicate the ideas upon which differential geometry is built.

We proceed as follows:

Let x be a point, x' another point near x. We write ∂x for the small quantity $|(xx')|$ and τx for $\overline{xx'}$.

Again, if ξ be a line, and ξ' another line near ξ, we write $\partial\xi$ for the small angle $(\xi\xi')$, and $p\xi$ for the point of intersection $\overline{\xi\xi'}$.

With these definitions we proceed to the differentiation of measures.

Thus we consider the values of

$$\frac{d\,|(x a)|}{dx},\quad \frac{d\,(x\alpha)}{dx},\quad \frac{d\,(\xi a)}{d\xi},\quad \frac{d\,(\xi\alpha)}{d\xi}.$$

We define, for instance, $\frac{d\,(x\alpha)}{dx}$ = Limit when x' tends to identity with x, of the expression

$$\frac{(x'\alpha)-(x\alpha)}{|(xx')|}.$$

It will be found that having differentiated the measures containing two elements, all the measures of more elements may be differentiated by a definite method, independent of the method of limits.

We next require the differentiation of determinates. It will be found that the differentiation of determinates may be reduced to the differentiation of measures.

The definition of the differential of a determinate, say $\overline{xa}$, is

$$\frac{d\overline{xa}}{dx} = \text{Limit when } x' \text{ tends to identity with } x \text{ of } \frac{(\overline{xa}\,\overline{x'a})}{|(xx')|}.$$

We consider next vectorial elements. First, we require the differentials of $a_{\rho\sigma\ \ \omega}$ and $a_{\rho\sigma\ .\omega\phi}$. Having found these we may find the values of the differentials of measures and determinates of vectorial elements. The same holds for equational elements.

These are all the formulae we require, and knowing these we may differentiate the most general measure and determinate. At the same time we may reduce measures containing the differential signs p; ν, τ.

CHAPTER I

FUNDAMENTALS OF THE GEOMETRY OF TWO, THREE AND FOUR ELEMENTS

§ 7. The measures containing two elements are

$$(ab),\ (a\beta),\ (\alpha\beta).$$

We suppose a line has "sense" as well as position. If α be a line, $\bar{\alpha}$ will be used to denote the line with same position, but with reversed sense.

We give a set of axioms, which we state as we require them.

As regards the interchange of elements we have

$$(ba) = -(ab) \quad \text{.........................(I)},$$
$$(\beta a) = (a\beta) \quad \text{.........................(II)},$$
$$(\beta\alpha) = -(\alpha\beta) \quad \text{.....................(III)}.$$

We have also the following axioms.

If a, b, c are three points incident in a line,

$$(bc) + (ca) + (ab) = 0 \quad \text{...................(IV)}.$$

The measure $(\alpha\bar{\alpha})$ is independent of α and equal to a constant positive quantity π ....(V).

α, β, γ being three lines

$$(\beta\gamma) + (\gamma\alpha) + (\alpha\beta) = 2\pi \quad \text{.................(VI)}.$$

Corollary. $\qquad (\alpha\bar{\beta}) = (\alpha\beta) + \pi.$

For $\qquad (\alpha\bar{\beta}) + (\bar{\beta}\beta) + (\beta\alpha) = 2\pi,$

$$\therefore\ (\alpha\bar{\beta}) = \pi + (\alpha\beta).$$

We have the following axiom for point and line,

$$(a\bar{\beta}) = -(a\beta) \quad \text{...................(VII)}.$$

Further we shall suppose $(a\beta)$ is positive when the sense of β in regard to a is counter-clockwise; and negative when clockwise.

As regards determinates we have the following:

$$\overline{ba} = \overline{a\bar{\bar{b}}} \quad \text{......................(VIII)},$$
$$\overline{\beta a} = \overline{a\beta} \quad \text{......................(IX)},$$
$$\overline{\alpha\bar{\beta}} = \overline{\alpha\beta} \quad \text{........................(X)}.$$

If $(\alpha c) = 0$, then $\qquad \overline{\overline{\alpha\beta} c} = \alpha$ or $\bar{\alpha}$ ....(XI).

If $(a\gamma) = 0$, then $\qquad \overline{\overline{ab}\gamma} = a$ ....(XII).

If a, b, c, d be four points incident in a line, and such that $\overline{ab} = \overline{cd}$, then $\dfrac{(ab)}{(cd)}$ is positive ....(XIII).

§ 8. *Geometry of two points and a line, a, b, γ. Definition of the sine function.*

We shall assume that the expression

$$\frac{(a\gamma) - (b\gamma)}{|(ab)|}$$

depends only on the line γ and the determinate $\overline{ab}$ ......(XIV).

We may therefore write it as a function of $(\overline{ab}\gamma)$. This function is the sine function. We have accordingly

$$\sin(\overline{ab}\gamma) = \frac{(a\gamma) - (b\gamma)}{|(ab)|}.$$

Corollary. $\qquad \sin(\overline{ba}\gamma) = \dfrac{(b\gamma) - (a\gamma)}{|(ab)|} = -\sin(\overline{ab}\gamma).$

Hence if α, β be two lines

$$\sin(\bar{\alpha}\beta) = -\sin(\alpha\beta),$$

also $\qquad \sin(\overline{ab}\bar{\gamma}) = \dfrac{(a\bar{\gamma}) - (b\bar{\gamma})}{|(ab)|} = -\dfrac{(a\gamma) - (b\gamma)}{|(ab)|}$

$$= -\sin(\overline{ab}\gamma).$$

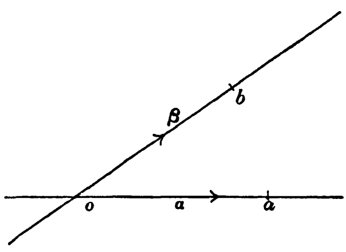
$$\therefore \sin(\alpha\bar{\beta}) = -\sin(\alpha\beta),$$

$$\therefore \sin\{(\alpha\beta) + \pi\} = -\sin(\alpha\beta).$$

Again, let α, β be two lines. Let $o = \overline{\alpha\beta}$, and let a, b be two points incident in α, β respectively, such that

$$|(oa)| = |(ob)|,$$

and $\qquad \overline{oa} = \alpha, \ \overline{ob} = \beta.$



Then $\sin(\alpha\beta) = \sin(\overline{oa}\beta)$

$$= -\frac{(a\beta)}{|(oa)|},$$

and $\sin(\beta\alpha) = \sin(\overline{ob}\alpha)$

$$= -\frac{(b\alpha)}{|(ob)|}.$$

Now from symmetry it is clear that a bears the same relatio. to β as b does to α; with the exception that the sense of α in regard to b is opposite to the sense of β in regard to a. Hence $(a\beta) = -(b\alpha)$.

Hence $$\sin(\beta\alpha) = -\sin(\alpha\beta),$$

or $$\sin\{-(\alpha\beta)\} = -\sin(\alpha\beta) \quad \text{.................(XV)}.$$

We now proceed to the Geometry of three and four elements. We shall first consider such geometry as involves only algebraic and sine functions; considering afterwards properties involving the cosine function as well.

§ 9. *Geometry of three points a, b, c.*

We write (abc)* for $|(ab)|(\overline{ab}c)$.

Hence $(bac) = |(ba)|(\overline{ba}c) = -|(ab)|(\overline{ab}c) = -(abc)$.

To find the value of (cab) we have

$$\sin(\overline{ab}\,\overline{ac}) = \frac{(a\overline{ac}) - (b\overline{ac})}{|(ab)|} \text{ by § 8}$$

$$= -\frac{(b\overline{ac})}{|(ab)|} = -\frac{(\overline{ac}b)}{|(ab)|}$$

$$= -\frac{(acb)}{|(ab)(ac)|},$$

similarly $$\sin(\overline{ac}\,\overline{ab}) = -\frac{(abc)}{|(ab)(ac)|}.$$

Now $$\sin(\overline{ac}\,\overline{ab}) = -\sin(\overline{ab}\,\overline{ac}),$$

$$\therefore (acb) = -(abc),$$

$$\therefore (cab) = (abc).$$

Hence $$(abc) = (bca) = (cab)$$

$$= -(bac) = -(cba) = -(acb),$$

also $$(abc) = |(ca)(ab)| \sin(\overline{ca}\,\overline{ab})$$

$$= |(ab)(bc)| \sin(\overline{ab}\,\overline{bc})$$

$$= |(bc)(ca)| \sin(\overline{bc}\,\overline{ca}).$$

Corollary. $$\frac{\sin(\overline{ca}\,\overline{ab})}{|(bc)|} = \frac{\sin(\overline{ab}\,\overline{bc})}{|(ca)|} = \frac{\sin(\overline{bc}\,\overline{ca})}{|(ab)|} = \frac{1}{2R}.$$

* (abc) as thus defined is equal to twice the area of the triangle formed by the points.

We shall call (abc) the standard measure of three points. R is called the circum-radius of the triangle.

§ 10. *Geometry of three lines α, β, γ.*

We write $(\alpha\beta\gamma)$ for $\sin(\alpha\beta) . (\overline{\alpha\beta}\gamma)$,

$\therefore\ (\beta\alpha\gamma) = \sin(\beta\alpha)(\overline{\beta\alpha}\gamma) = -\sin(\alpha\beta)(\overline{\alpha\beta}\gamma) = -(\alpha\beta\gamma)$.

Now if we put $a = \overline{\beta\gamma}$, $b = \overline{\gamma\alpha}$, $c = \overline{\alpha\beta}$; we get

$$\overline{bc} = \alpha,\ \overline{ca} = \beta,\ \overline{ab} = \gamma,$$

or

$$\overline{bc} = \bar{\alpha},\ \overline{ca} = \beta,\ \overline{ab} = \gamma,$$

or

$$\overline{bc} = \bar{\alpha},\ \overline{ca} = \bar{\beta},\ \overline{ab} = \gamma,$$

or

$$\overline{bc} = \bar{\alpha},\ \overline{ca} = \bar{\beta},\ \overline{ab} = \bar{\gamma},$$

or one other set of relations.

We shall consider only the first alternative; the same result follows from any one of them

We have $$\sin(\alpha\beta)(\overline{\alpha\beta}\gamma) = \sin(\overline{bc}\,\overline{ca})(c\overline{ab}) = \frac{(abc)^2}{|(bc)(ca)(ab)|}.$$

Hence $(\alpha\beta\gamma) = (\beta\gamma\alpha) = (\gamma\alpha\beta) = -(\beta\alpha\gamma) = -(\gamma\beta\alpha) = -(\alpha\gamma\beta)$,

and $(\alpha\beta\gamma) = (\overline{\alpha\beta}\gamma)\sin(\alpha\beta) =$ two similar expressions.

We shall call $(\alpha\beta\gamma)$ the standard measure of three lines.

It is easy to shew that

$$|(\alpha\beta\gamma)| = |(\overline{\gamma\alpha}\,\overline{\alpha\beta})\sin(\gamma\alpha)\sin(\alpha\beta)|$$
$$= \text{two similar expressions.}$$

§ 11. *Geometry of two points and two lines, a, b, γ, δ.*

To reduce the measure $(\overline{\overline{ab}\gamma}\delta)$.

Let $\overline{\overline{ab}\gamma} = o$.

Then $\overline{oa} = \overline{ob}$ or $\overline{oa} = \overline{\overline{ob}}$.

Suppose $\overline{oa} = \overline{ob}$, then

$$\sin(\overline{oa}\gamma) = -\frac{(a\gamma)}{|(oa)|} : \sin(\overline{ob}\gamma) = -\frac{(b\gamma)}{|(ob)|},$$

also

$$\sin(\overline{oa}\,\delta) = \frac{(o\delta) - (a\delta)}{|(oa)|} : \sin(\overline{ob}\,\delta) = \frac{(o\delta) - (b\delta)}{|(ob)|},$$

$$\therefore \frac{(a\gamma)}{(b\gamma)} = \frac{(o\delta) - (a\delta)}{(o\delta) - (b\delta)},$$

from which $$(o\delta) = \frac{(a\gamma)(b\delta) - (a\delta)(b\gamma)}{(a\gamma) - (b\gamma)},$$

$$\therefore\ (\overline{ab}\gamma\delta) = \frac{(a\gamma)(b\delta) - (a\delta)(b\gamma)}{(a\gamma) - (b\gamma)}.$$

Now $$(\overline{ab}\gamma\delta) = \sin(\overline{ab}\gamma)(\overline{\overline{ab}\gamma}\delta)$$
$$= \frac{(a\gamma)(b\delta) - (a\delta)(b\gamma)}{|(ab)|}.$$

It is easy to see that when $\overline{oa} = \overline{ob}$ the same result follows.

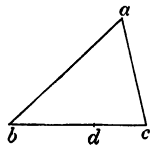
Writing $(ab\gamma\delta)$ for $|(ab)| \sin(\overline{ab}\gamma)(\overline{\overline{ab}\gamma}\delta)$,

we have $$(ab\gamma\delta) = (a\gamma)(b\delta) - (a\delta)(b\gamma).$$

We shall call $(ab\gamma\delta)$ the standard measure of two points and two lines.

§ 12. *Geometry of three lines and a point, α, β, γ, d.*

Let a, b, c be three points and d a point incident in $\overline{bc}$.



Then
$$(abd) + (adc) = |(bd)|(a\overline{bd}) + |(dc)|(a\overline{dc}).$$

Suppose $\overline{bd} = \overline{dc} = \overline{bc}$,

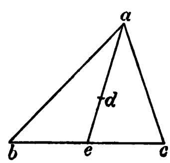
then $(a\overline{bd}) = (a\overline{dc}) = (a\overline{bc})$,

and $$|(bd)| + |(dc)| = |(bc)|,$$

$$\therefore\ (abd) + (adc) = |(bc)|(a\overline{bc}) = (abc).$$

The same result follows from the other alternatives to
$$\overline{bd} = \overline{dc} = \overline{bc}.$$

Now let d be any point, not necessarily incident in $\overline{bc}$.



Let $$\overline{\overline{ad}\,\overline{bc}} = e.$$

Then
$$(dbe) + (dec) = (dbc),$$
$$(abe) + (aec) = (abc),$$
$$(abe) = (abd) + (dbe),$$
$$(aec) = (adc) + (dec),$$

from which $$(dbc) + (dca) + (dab) = (abc).$$

Now let $$\overline{bc} = \alpha,\ \overline{ca} = \beta,\ \overline{ab} = \gamma.$$

Then $$(dbc) = |(bc)|\,(d\alpha), \text{ etc.}$$

$$\therefore\ |(bc)|\,(d\alpha) + |(ca)|\,(d\beta) + |(ab)|\,(d\gamma) = (abc),$$

$$\therefore\ \sin(\beta\gamma)\,(d\alpha) + \sin(\gamma\alpha)\,(d\beta) + \sin(\alpha\beta)\,(d\gamma) = (\alpha\beta\gamma) \text{ by § 9.}$$

The same result follows from any of the other alternatives of § 10; and is therefore true for any three lines.

§ 13. With regard to any line α we shall assume that one and only one line β, passing through a fixed point, can be found, so that $(\alpha\beta)$ has any given value, say θ. Further that all such lines through different points are parallel, i.e. the measure of any pair is zero ....(XVI).

Notation. We shall write α_θ for such a direction; so that

$$(\alpha\alpha_\theta) = \theta.$$

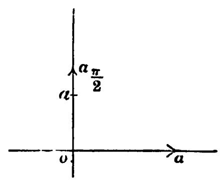
Corollary. $\alpha_{\frac{\pi}{2}\frac{\pi}{2}}$ is parallel to $\bar{\alpha}$.

Axiom. We shall assume that

$$|(\alpha\beta)| = |(\overline{\alpha_{\beta_{\frac{\pi}{2}}}\beta\,\alpha})| \quad \text{............ (XVII).}$$

To find the value of $\sin\frac{\pi}{2}$

Let $\overline{\alpha\alpha_{\frac{\pi}{2}}} = o$ and let a be a point incident in $\alpha_{\frac{\pi}{2}}$ such that $\overline{oa} = \alpha_{\frac{\pi}{2}}$.



Then

$$\sin(\alpha\alpha_{\frac{\pi}{2}}) = \sin(\alpha\overline{oa}) = -\sin(\overline{oa}\alpha)$$

$$= \frac{(a\alpha)}{|(\overline{oa})|}$$

$$= \frac{(a\alpha)}{|(\overline{a a_{\alpha_{\frac{\pi}{2}}}}\alpha)|} = \frac{(a\alpha)}{|(a\alpha)|}$$

$= +1$ since the orientation of α in regard to a is counter-clockwise ....(XVIII).

§ 14. *Definition of cosine function.*

We define the cosine of the measure $(\alpha\beta)$ as follows:

$$\cos(\alpha\beta) = \sin(\alpha\beta_{\frac{\pi}{2}}).$$

Now $$(\alpha\beta_{\frac{\pi}{2}}) + (\beta_{\frac{\pi}{2}}\beta) + (\beta\alpha) = 2\pi,$$

$$\therefore\ (\alpha\beta_{\frac{\pi}{2}}) = 2\pi + (\alpha\beta) + \frac{\pi}{2}.$$

Now

$$\sin(\alpha\beta) = -\sin\{(\alpha\beta)+\pi\} = \sin\{-\pi-(\alpha\beta)\} = \sin\{\pi-(\alpha\beta)\},$$

$$\therefore \sin\left\{(\alpha\beta)+\frac{\pi}{2}\right\} = \sin\left\{\frac{\pi}{2}-(\alpha\beta)\right\}.$$

Again,
$$(\alpha_{\frac{\pi}{2}}\beta)+(\beta\alpha)+(\alpha\alpha_{\frac{\pi}{2}}) = 2\pi,$$

$$\therefore (\alpha_{\frac{\pi}{2}}\beta) = 2\pi+(\alpha\beta)-\frac{\pi}{2},$$

$$\therefore \sin(\alpha_{\frac{\pi}{2}}\beta) = \sin\left\{\frac{\pi}{2}+(\alpha\beta)\right\} = -\sin\left\{\frac{\pi}{2}-(\alpha\beta)\right\}$$

$$= -\sin\left\{\frac{\pi}{2}+(\alpha\beta)\right\} = -\cos(\alpha\beta),$$

$$\therefore \cos(\alpha\beta) = \sin(\alpha\beta_{\frac{\pi}{2}}) = -\sin(\alpha_{\frac{\pi}{2}}\beta)$$

$$= \sin\left\{\frac{\pi}{2}+(\alpha\beta)\right\} = \sin\left\{\frac{\pi}{2}-(\alpha\beta)\right\}.$$

Again,

$$\cos(\beta\alpha) = \sin\left\{\frac{\pi}{2}+(\beta\alpha)\right\} = \sin\left\{\frac{\pi}{2}-(\alpha\beta)\right\} = \cos(\alpha\beta),$$

$$\cos(\alpha_{\frac{\pi}{2}}\beta) = \cos\left\{\frac{\pi}{2}-(\alpha\beta)\right\} = \sin(\alpha\beta),$$

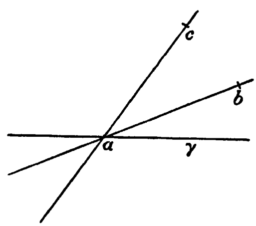
$$\cos(\alpha\beta_{\frac{\pi}{2}}) = \cos\left\{\frac{\pi}{2}+(\alpha\beta)\right\} = -\sin(\alpha\beta),$$

$$\cos\{\pi-(\alpha\beta)\} = -\cos(\alpha\beta),$$

$$\cos\{\pi+(\alpha\beta)\} = -\cos(\alpha\beta),$$

$$\cos(\alpha\bar{\beta}) = \cos(\bar{\alpha}\beta) = -\cos(\alpha\beta).$$

§ 15. *Addition formulae for sine and cosine functions.*



The three lines $\overline{ab}$, $\overline{ac}$, γ where $(a\gamma)=0$, denote three arbitrary directions.

Now $(\overline{ab}\gamma)-(\overline{ac}\gamma) = 2\pi+(\overline{ab}\,\overline{ac})$.

Now
$$\sin(\overline{ab}\gamma)\cos(\overline{ac}\gamma)-\sin(\overline{ac}\gamma)\cos(\overline{ab}\gamma)$$
$$=\sin(\overline{ab}\gamma)\sin(\overline{ac}\gamma_{\frac{\pi}{2}})-\sin(\overline{ac}\gamma)\sin(\overline{ab}\gamma_{\frac{\pi}{2}})$$
$$=\frac{(b\gamma)}{|(ab)|}\frac{(c\gamma_{\frac{\pi}{2}})}{|(ac)|}-\frac{(c\gamma)}{|(ac)|}\frac{(b\gamma_{\frac{\pi}{2}})}{|(ab)|},\ \text{supposing } (a\gamma_{\frac{\pi}{2}})=0$$
$$=\frac{(bc\gamma\gamma_{\frac{\pi}{2}})}{|(ab)(ac)|}\ \text{by formula on p. 12}$$
$$=\frac{|(bc)|\sin(\gamma\gamma_{\frac{\pi}{2}})(\overline{bc}a)}{|(ab)(ac)|}$$
$$=\frac{(abc)}{|(ab)(ca)|}=\sin(\overline{ca}\,ab)$$
$$=\sin(\overline{ab}\,\overline{ac})=\sin\{(\overline{ab}\gamma)-(\overline{ac}\gamma)\}.$$

If we put $(\overline{ab}\gamma)=\theta$, $(\overline{ac}\gamma)=\phi$, this becomes
$$\sin(\theta-\phi)=\sin\theta\cos\phi-\sin\phi\cos\theta,$$
and we have the other trigonometric formulae.

§ 16. *Further geometry of the triangle* $\left.\begin{matrix}abc\\ \alpha\beta\gamma\end{matrix}\right\}$.

In this notation $\alpha=\overline{bc}$, $\beta=\overline{ca}$, $\gamma=\overline{ab}$.

We have
$$(\beta\gamma)+(\gamma\alpha)+(\alpha\beta)=2\pi,$$
$$\therefore\ -\sin(\alpha\beta)=\sin\{(\beta\gamma)+(\gamma\alpha)\}$$
$$=\sin(\beta\gamma)\cos(\gamma\alpha)+\sin(\gamma\alpha)\cos(\beta\gamma).$$

Hence
$$-|(ab)|=|(bc)|\cos(\gamma\alpha)+|(ca)|\cos(\beta\gamma),$$
similarly
$$-|(ca)|=|(ab)|\cos(\beta\gamma)+|(bc)|\cos(\alpha\beta)$$
$$-|(bc)|=|(ca)|\cos(\alpha\beta)+|(ab)|\cos(\gamma\alpha).$$

From which
$$(bc)^2=(ca)^2+(ab)^2+2\,|(ca)(ab)|\cos(\overline{ca}\,ab),$$
and two similar formulae.

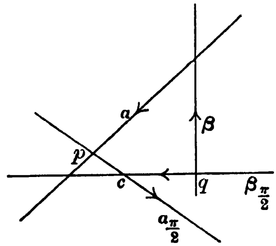
We may now reduce the standard measure of three points.

It may be shewn that
$$4(abc)^2=2(ca)^2(ab)^2+2(ab)^2(bc)^2+2(bc)^2(ca)^2$$
$$-(bc)^4-(ca)^4-(ab)^4.$$

§ 17. *Geometry of two lines and a point, α, β, c.*

To reduce $(\overline{\alpha\beta c})^2$.

Let $p = \overline{\alpha c_{\alpha_{\frac{\pi}{2}}}}$, $q = \overline{\beta c_{\beta_{\frac{\pi}{2}}}}$.



Now in the triangle $\overline{pq}$, β, α

$$\frac{|(\overline{pq}\,\beta\,\overline{\beta\alpha})|}{\sin(\alpha\overline{pq})} = \frac{(\overline{\beta\alpha}\,\alpha\overline{pq})|}{\sin(\overline{pq}\beta)} = \frac{|(\alpha\,\overline{pq}\,\overline{pq}\beta)|}{\sin(\beta\alpha)},$$

$$\therefore \frac{|(\overline{\alpha\beta}q)}{\sin(\overline{pq}\alpha)} = \frac{|(pq)|}{\sin(\alpha\beta)}.$$

Again, we have

$$\frac{\sin(\overline{cp}\,\overline{pq})}{|(cq)|} = \frac{\sin(\overline{qc}\,\overline{cp})}{|(pq)|} = \frac{\sin(\beta\bar{\alpha})}{|(pq)|} = \frac{\sin(\alpha\beta)}{|(pq)|}.$$

$$\therefore \frac{\cos(\alpha\overline{pq})}{|(cq)|} = \frac{\sin(\overline{pq}\alpha)}{|(\overline{\alpha\beta}q)|},$$

$$\therefore \tan(\overline{pq}\,\alpha) = \frac{|(\overline{\alpha\beta}q)|}{|(cq)|} = \tan(\overline{cq}\,c\overline{\alpha\beta}),$$

$$\therefore \sin(\overline{pq}\alpha) = \pm\sin(\overline{cq}\,c\overline{\alpha\beta}).$$

Hence
$$\frac{(pq)^2}{\sin^2(\alpha\beta)} = \frac{(\overline{\alpha\beta}q)^2}{\sin^2(\overline{cq}\,c\overline{\alpha\beta})} = (\overline{\alpha\beta}c)^2.$$

Firstly, suppose the sense of α, β to be counter-clockwise in regard to c,

$$\begin{aligned}\therefore (\overline{\alpha\beta}c)^2\sin^2(\alpha\beta) &= (pq)^2 \\ &= (pc)^2 + (qc)^2 - 2\,|(pc)(qc)|\cos(\overline{pc}\,\overline{qc}) \\ &= (pc)^2 + (qc)^2 - 2\,|(pc)(qc)|\cos(\alpha_{\frac{\pi}{2}}\beta_{\frac{\pi}{2}}) \\ &= (c\alpha)^2 + (c\beta)^2 - 2\,(c\alpha)(c\beta)\cos(\alpha\beta),\end{aligned}$$

since $(c\alpha)$, $(c\beta)$ are both positive.

Secondly, suppose the sense of α to be counter-clockwise in regard to c, while that of β is clockwise.

Then the senses of α, $\bar{\beta}$ are both counter-clockwise in regard to c.

Hence $(\overline{\alpha\bar{\beta}}c)^2\sin^2(\alpha\bar{\beta}) = (c\alpha)^2 + (c\bar{\beta})^2 - 2\,(c\alpha)(c\bar{\beta})\cos(\alpha\bar{\beta})$,

$$\therefore (\overline{\alpha\beta}c)^2\sin^2(\alpha\beta) = (c\alpha)^2 + (c\beta)^2 - 2\,(c\alpha)(c\beta)\cos(\alpha\beta).$$

Similarly in the case when both α, β have senses which are clockwise in regard to c, we obtain the same result.

Hence in all cases

$$\sin^2(\alpha\beta)(\overline{\alpha\beta}c)^2 = (c\alpha)^2 + (c\beta)^2 - 2\,(c\alpha)(c\beta)\cos(\alpha\beta).$$

Hence $\sin(\alpha\beta)\,|(\overline{\alpha\beta}c)| = \sqrt{(c\alpha)^2 + (c\beta)^2 - 2(c\alpha)(c\beta)\cos(\alpha\beta)}$,

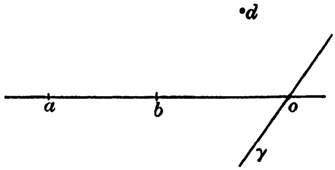
where the square root has the sign of $(\alpha\beta)$.

The quantity $(\alpha\beta c) = \sin(\alpha\beta)\,|(\overline{\alpha\beta}c)|$ we shall call the standard measure of two lines and a point.

§ 18. *Geometry of three points and a line, a, b, γ, d.*

To find the value of $(\overline{\overline{ab}\gamma}d)^2$.

Let $\overline{\overline{ab}\gamma} = o$.



In the triangle whose vertices are a, o, d we have

$$(ad)^2 = (ao)^2 + (od)^2 + 2\,|(ao)(od)|\cos(\overline{ao}\,\overline{od})$$

Similarly $(bd)^2 = (bo)^2 + (od)^2 + 2\,|(bo)(od)|\cos(\overline{bo}\,\overline{od})$.

We shall consider only the case in which $\overline{ao} = \overline{bo} = \overline{ab}$; in the other cases, the theorem can be proved in a similar manner.

Multiplying the first equation by $|(bo)|$ and the second by $|(ao)|$ and subtracting, we have

$$(ad)^2\,|(bo)| - (bd)^2\,|(ao)| = (ao)^2\,|(bo)| - (bo)^2\,|(ao)| + (od)^2\,\{|(bo)| - |(ao)|\},$$

$$\therefore\ (od)^2 = \frac{(ad)^2\,|(bo)| - (bd)^2\,|(ao)|}{-\,|(ab)|} + |(ao)(bo)|.$$

Now
$$\sin(\overline{oa}\gamma) = \sin(\overline{ob}\gamma),$$

$$\therefore\ \frac{(a\gamma)}{|(oa)|} = \frac{(b\gamma)}{|(ob)|};$$

also
$$|(oa)| - |(ob)| = |(ab)|,$$

$$\therefore\ \frac{(a\gamma)}{|(oa)|} = \frac{(b\gamma)}{|(ob)|} = \frac{(a\gamma)-(b\gamma)}{|(ab)|},$$

$$\therefore\ |(oa)| = \frac{(a\gamma)}{(a\gamma)-(b\gamma)}\,|(ab)|,$$

$$|(ob)| = \frac{(b\gamma)}{(a\gamma)-(b\gamma)}\,|(ab)|,$$

$$\therefore\ (\overline{\overline{ab}\gamma}d)^2 = \frac{(bd)^2(a\gamma)-(ad)^2(b\gamma)}{(a\gamma)-(b\gamma)} + \frac{(a\gamma)(b\gamma)}{\{(a\gamma)-(b\gamma)\}^2}(ab)^2.$$

The quantity $(ab\gamma d) = |(ab)| \sin(\overline{ab}\gamma)(\overline{\overline{ab}\gamma d})$ we shall call the standard measure of the elements a, b, γ, d

$$(ab\gamma d)^2 = \{(bd)^2(a\gamma) - (ad)^2(b\gamma)\}\{(a\gamma) - (b\gamma)\} + (a\gamma)(b\gamma)(ab)^2$$
$$= (bd)^2(a\gamma)^2 + (ad)^2(b\gamma)^2 + (a\gamma)(b\gamma)\{(ab)^2 - (ad)^2 - (bd)^2\}.$$

§ 19. *Further geometry of two lines and two points, α, β, c, d.*

To find the value of $(\overline{\overline{\alpha\beta}cd})$.

$$(\overline{\overline{\alpha\beta}cd}) = \frac{(\alpha\beta cd)}{|(\overline{\alpha\beta}c)|} = \frac{(cd\overline{\alpha\beta})}{|(\overline{\alpha\beta}c)|}$$
$$= \frac{(cd\alpha\beta)}{|(\overline{\alpha\beta}c)|\sin(\alpha\beta)}$$
$$= \frac{(c\alpha)(d\beta) - (c\beta)(d\alpha)}{\sqrt{(c\alpha)^2 + (c\beta)^2 - 2(c\alpha)(c\beta)\cos(\alpha\beta)}},$$

where the square root has the sign of $(\alpha\beta)$.

If $$(\alpha\beta cd) = (\alpha\beta c)(\overline{\overline{\alpha\beta}cd}),$$
it is clear that $$(\alpha\beta cd) = (cd\alpha\beta).$$

§ 20. *Geometry of three lines and a point, α, β, c, δ.*

To find the value of $\sin(\overline{\overline{\alpha\beta}c}\delta)$.

$$\sin(\overline{\overline{\alpha\beta}c}\delta) = \frac{(\overline{\alpha\beta}\delta) - (c\delta)}{|(\overline{\alpha\beta}c)|}$$
$$= \frac{(\alpha\beta\delta) - (c\delta)\sin(\alpha\beta)}{(\alpha\beta c)}.$$

Now $$(\alpha\beta\delta) = (c\alpha)\sin(\beta\delta) + (c\beta)\sin(\delta\alpha) + (c\delta)\sin(\alpha\beta),$$

$$\therefore \sin(\overline{\overline{\alpha\beta}c}\delta) = \frac{(c\alpha)\sin(\beta\delta) + (c\beta)\sin(\delta\alpha)}{(\alpha\beta c)}$$
$$= \frac{(\alpha c)\sin(\beta\delta) - (\beta c)\sin(\alpha\delta)}{(\alpha\beta c)}.$$

We shall call $$(\alpha\beta c\delta) = (\alpha\beta c)\sin(\overline{\overline{\alpha\beta}c}\delta)$$
the standard measure of α, β, c, δ, so that

$$(\alpha\beta c\delta) = (\alpha c)\sin(\beta\delta) - (\beta c)\sin(\alpha\delta).$$

§ 21. *Eliminants.*

Let S be an arbitrary set of elements. Then a relation between the measure of pairs of elements selected from this set we shall call an eliminant of the set.

We have for three lines

$$(\beta\gamma)+(\gamma\alpha)+(\alpha\beta)=2\pi.$$

For four elements there is with one exception an eliminant between the six measures of the six pairs of elements we can get from the four elements.

(i) *Four points a, b, c, d.*

We have, see Casey's *Analytical Geometry*, p. 305, formula (756),

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & (ab)^2 & (ac)^2 & (ad)^2 \\ 1 & (ba)^2 & 0 & (bc)^2 & (bd)^2 \\ 1 & (ca)^2 & (cb)^2 & 0 & (cd)^2 \\ 1 & (da)^2 & (db)^2 & (dc)^2 & 0 \end{vmatrix}=0.$$

If this be expanded and reduced by the relation

$$(bc)^2=(ca)^2+(ab)^2+2\,|(ca)(ab)|\cos(\overline{ca}\,\overline{ab}),$$

we get $\Sigma\,(bc)^2(ad)^4+2\Sigma\,|(ca)(ab)|\cos(\overline{ca}\,\overline{ab})\,(bd)^2(cd)^2$

$$+\,2\Sigma\,|(bc)(ca)(ab)|\,\Sigma\,|(bc)|\cos(\overline{ca}\,\overline{ab})\,(ad)^2+(bc)^2(ca)^2(ab)^2=0.$$

(ii) *For three points and a line, a, b, c, δ.*

Now if $\qquad \theta+\phi+\psi=2\pi,$

then $\quad 1-\cos^2\theta-\cos^2\phi-\cos^2\psi+2\cos\theta\cos\phi\cos\psi=0.$

Now $\qquad (\overline{bc}\,\delta_{\frac{\pi}{2}})+(\delta_{\frac{\pi}{2}}\overline{ca})+(\overline{ca}\,\overline{bc})=2\pi,$

$$\therefore\ 1-\sin^2(\overline{bc}\,\delta)-\sin^2(\delta\,\overline{ca})-\cos^2(\overline{ca}\,\overline{bc})$$
$$-\,2\cos(\overline{ca}\,\overline{bc})\sin(\delta\,\overline{ca})\sin(\overline{bc}\,\delta)=0,$$

$$\therefore\ \frac{\{(b\delta)-(c\delta)\}^2}{(bc)^2}+\frac{\{(c\delta)-(a\delta)\}^2}{(ca)^2}+2\,\frac{\{(b\delta)-(c\delta)\}\{(c\delta)-(a\delta)\}}{|(bc)(ca)|}\cos(\overline{bc}\,\overline{ca})$$
$$=\sin^2(\overline{bc}\,\overline{ca}),$$

$$\therefore\ (ca)^2\{(b\delta)-(c\delta)\}^2+(bc)^2\{(c\delta)-(a\delta)\}^2$$
$$+\,2\,|(bc)(ca)|\cos(\overline{bc}\,\overline{ca})\,\{(b\delta)-(c\delta)\}\,\{(c\delta)-(a\delta)\}=(abc)^2.$$

$$\therefore\ (a\delta)^2(bc)^2+(b\delta)^2(ca)^2+(c\delta)^2(ab)^2$$
$$-\,2\,|(ca)(ab)|\cos(\overline{ca}\,\overline{ab})\,.\,(b\delta)(c\delta)-2\,|(ab)(bc)|\cos(\overline{ab}\,\overline{bc})\,.\,(c\delta)(a\delta)$$
$$-\,2\,|(bc)(ca)|\cos(\overline{bc}\,\overline{ca})\,.\,(a\delta)(b\delta)=(abc)^2.$$

(iii) *For two points and two lines, a, b, γ, δ.*

We have

$$\sin^2(\gamma\delta)\,(\overline{a_\gamma a_\delta b})^2=(ba_\gamma)^2+(ba_\delta)^2-2\,(ba_\gamma)(ba_\delta)\cos(\gamma\delta),$$

where a_γ denotes the line through a parallel to γ.

Now $(ba_\gamma) = (ba_\gamma) - (aa_\gamma) = (ba\,a_\gamma) = |(ba)| \sin(\overline{ba}\,a_\gamma)$

$$= |(ba)| \sin(\overline{ba}\gamma) = (ba\gamma),$$

$$\therefore\ \sin^2(\gamma\delta)(ab)^2 = \{(b\gamma) - (a\gamma)\}^2 + \{(b\delta) - (a\delta)\}^2 - 2\{(b\gamma) - (a\gamma)\}\{(b\delta) - (a\delta)\}\cos(\gamma\delta).$$

(iv) *For three lines and a point, α, β, γ, d.*

We have no eliminant in this case.

§ 22. *Examples.*

1. Shew that if a, β, γ, δ be four lines

$$\sin(\beta\gamma)\sin(a\delta) + \sin(\gamma a)\sin(\beta\delta) + \sin(a\beta)\sin(\gamma\delta) = 0.$$

To prove this, we have

$$(\beta\delta) + (\delta a) + (a\beta) = 2\pi, \quad \therefore\ (\beta\delta) = 2\pi + (a\delta) - (a\beta),$$

$$(\gamma\delta) + (\delta a) + (a\gamma) = 2\pi, \qquad (\gamma\delta) = 2\pi + (a\delta) + (\gamma a),$$

$$\therefore\ \sin(\beta\delta) = \sin(a\delta)\cos(a\beta) - \cos(a\delta)\sin(a\beta),$$

$$\sin(\gamma\delta) = \sin(a\delta)\cos(\gamma a) + \cos(a\delta)\sin(\gamma a),$$

and hence we have the above formula.

2. Shew also that

$$\sin(\beta\gamma)\sin(\beta\delta)\sin(\gamma\delta) + \sin(\gamma a)\sin(\gamma\delta)\sin(a\delta) + \sin(a\beta)\sin(a\delta)\sin(\beta\delta) + \sin(\beta\gamma)\sin(\gamma a)\sin(a\beta) = 0.$$

This may be proved in a similar manner.

3. Shew that $(\overline{xy\zeta}a)^2 = \dfrac{(ya)^2 - k\{(xa)^2 + (ya)^2 - (xy)^2\} + k^2(xa)^2}{(1-k)^2}$,

where $$k = \frac{(y\zeta)}{(x\zeta)},$$

and that when k is small,

$$|(\overline{xy\zeta}a)| = |(ya)| + k\,|(xy)|\cos(\overline{yx}\,\overline{ya}).$$

The first is derivable from the formula

$$(\overline{ab\delta}c)^2 = \frac{(a\delta)(bc)^2 - (b\delta)(ac)^2}{(a\delta) - (b\delta)} + \frac{(a\delta)(b\delta)(ab)^2}{\{(a\delta) - (b\delta)\}^2}.$$

If k be small, $$|(\overline{xy\zeta}a)| = \left|\frac{(ya)\left\{1 + \dfrac{k\{(xy)^2 - (xa)^2 - (ya)^2\}}{(ya)^2}\right\}^{\frac{1}{2}}}{1-k}\right|$$

$$= \left[|(ya)| + \frac{k\{(xy)^2 - (xa)^2 - (ya)^2\}}{2|(ya)|}\right](1+k)$$

$$= |(ya)| + \frac{k\,(xy)^2 + (ya)^2 - (xa)^2}{2|(ya)|}$$

$$= |(ya)| - k\,|(xy)|\cos(\overline{xy}\,\overline{ya}).$$

4 Shew that $(\overline{xy\zeta}a) = \dfrac{(ya) - k(xa)}{1-k}$, where $k = \dfrac{(y\zeta)}{(x\zeta)}$

$$= (ya) + k\{(ya) - (xa)\}, \text{ when } k \text{ is small.}$$

These are derivable from the formula

$$(\overline{ab\gamma\delta})=\frac{(a\gamma)(b\delta)-(b\gamma)(a\delta)}{(a\gamma)-(b\gamma)}.$$

5. Shew that
$$(\overline{\xi\eta za})=\frac{(\eta a)-k(\xi a)}{\sqrt{1+2k\cos(\xi\eta)+k^2}},$$

where
$$k=\frac{(\eta z)}{(\xi z)}$$

and where the square root has the sign of $(\xi\eta)$,

$$=\pm\{(\eta a)-k(b\xi)\},$$

when k is small and where the sign is that of $(\xi\eta)$ and b is the foot of the perpendicular from a on η.

We have the formula

$$(\overline{a\beta cd})=\frac{(ac)(\beta d)-(ad)(\beta c)}{\sqrt{(ac)^2+(\beta c)^2-2(ac)(\beta c)\cos(a\beta)}},$$

where the square root has the sign of $(a\beta)$.

$$\therefore\ (\overline{\xi\eta za})=\frac{(\xi z)(\eta a)-(\eta z)(\xi a)}{\sqrt{(\xi z)^2+(\eta z)^2-2(\xi z)(\eta z)\cos(\xi\eta)}}$$
$$=\frac{(\eta a)-k(\xi a)}{\sqrt{1-2k\cos(\xi\eta)+k^2}}.$$

First, suppose $(\xi\eta)$ is positive:

$$(\overline{\xi\eta za})=\frac{(\eta a)-k(\xi a)}{|\sqrt{1-2k\cos(\xi\eta)+k^2}|}$$
$$=\{(\eta a)-k(\xi a)\}\{1+k\cos(\xi\eta)\},\quad k \text{ small},$$
$$=(\eta a)+k\{(\eta a)\cos(\xi\eta)-(\xi a)\}.$$

Secondly, suppose $(\xi\eta)$ is negative:

$$(\overline{\xi\eta za})=\frac{(\eta a)-k(\xi a)}{-|\sqrt{1-2k\cos(\xi\eta)+k^2}|}$$
$$=-(\eta a)-k\{(\eta a)\cos(\xi\eta)-(\xi a)\}$$

6. Shew that

$$\cos(\overline{\xi\eta za})=\frac{\cos(\eta a)-k\cos(\xi a)}{\sqrt{1-2k\cos(\xi\eta)+k^2}},\quad \sin(\overline{\xi\eta za})=\frac{\sin(\eta a)-k\sin(\xi a)}{\sqrt{1-2k\cos(\xi\eta)+k^2}},$$

where the square root has the sign of $(\xi\eta)$.

If k be small, shew that $(\overline{\xi\eta za})=\pm\{(\eta a)-k\sin(\xi\eta)\}$, according to the sign of $(\xi\eta)$.

The first two formulae are obtained from the formulae on p. 18.

When k is small, firstly $(\xi\eta)+$,

$$\begin{aligned}\sin(\overline{\overline{\xi\eta}za}) &= \{\sin(\eta a) - k\sin(\xi a)\}\{1 + k\cos(\xi\eta)\} \\ &= \sin(\eta a) - k\{\sin(\xi a) - \sin(\eta a)\cos(\xi\eta)\} \\ &= \sin(\eta a) - k\cos(\eta a)\sin(\xi\eta), \\ \therefore\ (\overline{\overline{\xi\eta}za}) &= \sin^{-1}\{\sin(\eta a) - k\cos(\eta a)\sin(\xi\eta)\} \\ &= (\eta a) - k\cos(\eta a)\sin(\xi\eta)\frac{1}{\sqrt{1-\sin^2(\eta a)}} \\ &= (\eta a) - k\sin(\xi\eta).\end{aligned}$$

Secondly $(\xi\eta)-$,

$$(\overline{\overline{\xi\eta}za}) = -(\eta a) + k\sin(\xi\eta).$$

7. Shew that

$$|(aa')(bb')(cc')|(\overline{aa'}\,\overline{bb'}\,\overline{cc'}) = (abb')(a'cc') - (a'bb')(acc').$$

Let $$\overline{bb'} = \beta,\quad \overline{cc'} = \gamma;$$

then $$(\overline{aa'}\beta\gamma) = \frac{(a\beta)(a'\gamma) - (a'\beta)(a\gamma)}{|(aa')|},$$

$$\therefore\ |(aa')|(\overline{aa'}\,\overline{bb'}\,\overline{cc'}) = (a\overline{bb'})(a'\overline{cc'}) - (a\overline{cc'})(a'\overline{bb'}).$$

Hence the required result

Similarly,

$$\sin(aa')\sin(\beta\beta')\sin(\gamma\gamma')(\overline{aa'}\,\overline{\beta\beta'}\,\overline{\gamma\gamma'}) = (a\beta\beta')(a'\gamma\gamma') - (a'\beta\beta')(a\gamma\gamma').$$

8. If $\left.\begin{matrix}abc\\ \alpha\beta\gamma\end{matrix}\right\},\ \left.\begin{matrix}a'b'c'\\ \alpha'\beta'\gamma'\end{matrix}\right\}$ denote two triangles, shew that

$$|(aa')(bb')(cc')|(\overline{aa'}\,\overline{bb'}\,\overline{cc'}) = 4RR'\sin(aa')\sin(\beta\beta')\sin(\gamma\gamma')(\overline{aa'}\,\overline{\beta\beta'}\,\overline{\gamma\gamma'}),$$

where R, R' are the circum-radii of the triangles.

$$\begin{aligned}&|(aa')(bb')(cc')|(\overline{aa'}\,\overline{bb'}\,\overline{cc'}) \\ &\quad= (abb')(a'cc') - (acc')(a'bb') \\ &\quad= (\overline{\beta\gamma}\,\overline{\gamma a}\,\overline{\gamma'a'})(\overline{\beta'\gamma'}\,\overline{a\beta}\,\overline{a'\beta'}) - (\overline{\beta'\gamma'}\,\overline{\gamma a}\,\overline{\gamma'a'})(\overline{\beta\gamma}\,\overline{a\beta}\,\overline{a'\beta'}) \\ &\quad= \frac{(\beta\gamma a)(\gamma\gamma'a')(\beta'a\beta)(\gamma'a'\beta') - (\beta'\gamma'a')(\gamma'\gamma a)(\beta a'\beta')(\gamma a\beta)}{\sin(\beta\gamma)\sin(\gamma a)\sin(a\beta)\sin(\beta'\gamma')\sin(\gamma'a')\sin(a'\beta')} \\ &\quad= \frac{(a\beta\gamma)(a'\beta'\gamma')}{\Pi\sin(\beta\gamma)\,\Pi\sin(\beta'\gamma')}[(a'\gamma\gamma')(a\beta\beta') - (a\gamma\gamma)(a'\beta\beta')] \\ &\quad= 4RR'\sin(aa')\sin(\beta\beta')\sin(\gamma\gamma')(\overline{aa'}\,\overline{\beta\beta'}\,\overline{\gamma\gamma'}).\end{aligned}$$

9. Shew that

$$(a\lambda)(mbc) + (b\lambda)(mca) + (c\lambda)(mab) = (m\lambda)(abc),$$

$$(ma)(\lambda\beta\gamma) + (m\beta)(\lambda\gamma a) + (m\gamma)(\lambda a\beta) = (m\lambda)(a\beta\gamma).$$

$$\begin{aligned}&(ma)(\lambda\beta\gamma) + (m\beta)(\lambda\gamma a) + (m\gamma)(\lambda a\beta) \\ &\quad= (ma)\{(m\lambda)\sin(\beta\gamma) + (m\beta)\sin(\gamma\lambda) - (m\gamma)\sin(\beta\lambda)\} + \ldots + \ldots \\ &\quad= (ma)(m\lambda)\sin(\beta\gamma) + \ldots + \ldots \\ &\quad= (m\lambda)(a\beta\gamma).\end{aligned}$$

The first relation is virtually the same as the second.

10. Shew that $$(\alpha\beta c\delta\lambda) = -(\lambda\delta c\beta\alpha).$$

Now
$$\begin{aligned}(\alpha\beta c\delta\lambda) &= (\overline{\alpha\beta}\, c\delta\lambda)\sin(\alpha\beta)\\ &= (\lambda\delta c\,\overline{\alpha\beta})\sin(\alpha\beta)\\ &= (\overline{\overline{\lambda\delta}\, c}\;\overline{\alpha\beta})\,(\lambda\delta c)\sin(\alpha\beta)\\ &= (\overline{\overline{\lambda\delta}\, c}\,\alpha\beta)\,(\lambda\delta c)\\ &= (\overline{\overline{\overline{\lambda\delta}\, c}\,\alpha}\,\beta)\,(\lambda\delta c\alpha)\\ &= (\lambda\delta c\alpha\beta) = -(\lambda\delta c\beta\alpha).\end{aligned}$$

11. Shew that $$(ab\gamma d\lambda\mu) = (\mu\lambda d\gamma ba).$$
$$\begin{aligned}(ab\gamma d\lambda\mu) &= (\overline{ab}\,\gamma d\lambda\mu)\,|(ab)|\\ &= -(\mu\lambda d\gamma\,\overline{ab})\,|(ab)|\\ &= (\mu\lambda d\gamma ba).\end{aligned}$$

12. Similarly, shew that in the case of seventh order measures
$$(ab\gamma d\lambda mn) = -(nm\lambda d\gamma ba),$$
$$(\alpha\beta c\delta l\mu\nu) = -(\nu\mu l\delta c\beta\alpha).$$

13. If $(\alpha\beta c\delta f) = \sin(\alpha\beta)\,(\overline{\alpha\beta}\,c\delta f)$, shew that
$$\begin{aligned}(\alpha\beta c\delta f)^2 &= (\alpha c)^2(\beta\delta f)^2 + (\beta c)^2(\alpha\delta f)^2\\ &\quad + 2(\alpha c)(\beta c)\{(\beta f)(\delta f)\cos(\alpha\delta) + (\alpha f)(\delta f)\cos(\beta\delta) - (\alpha f)(\beta f) - (\delta f)^2\cos(\alpha\beta)\}.\end{aligned}$$

CHAPTER II

REDUCTION OF MEASURES CONTAINING VECTORIAL ELEMENTS

§ 23. The general vectorial point is denoted by $a_{\acute\rho\acute\sigma\ \ldots\ \acute\omega}$, where a is a point, and $\acute\rho$, $\acute\sigma \ldots \acute\omega$ vectors.

The direction of $\acute\rho$ is ρ, and the magnitude is denoted by $\hat\rho$. It may seem convenient to regard $\hat\rho$ as always positive, and measure it in the direction of ρ. There is however an alternative convention, which proves to be more comprehensive. This is to regard $\hat\rho$ as positive or negative and to measure it in the direction of ρ when positive, and to measure it in the direction of $\bar\rho$ when negative. We shall also use an alternative notation to $a_{\acute\rho}$, namely $a_{\rho,\hat\rho}$, so that with the convention stated

$$a_{\rho,\hat\rho} = a_{\bar\rho,-\hat\rho}.$$

We may see the use of the convention when expressing the foot of the perpendicular from a on β by means of a vector.

It is
$$a_{\beta\frac{\pi}{2},\,-(a\beta)} \text{ or } a_{\beta_{-\frac{\pi}{2}},\,(a\beta)}.$$

With the restricted convention we are not able to represent it by means of one formula.

§ 24. *To express* $(a_{\acute\rho\acute\sigma\ \ldots\ \acute\omega}b)^2$ *in terms of measures of two elements and vectorial magnitudes.*

Let
$$a_{\acute\rho} = c.$$

First, suppose $\hat\rho$ positive, then

$$|(ac)| = \hat\rho,\ \overline{ac} = \rho.$$

Now
$$(bc)^2 = (ab)^2 + (ac)^2 + 2\,|(ab)\,(ac)|\cos(\overline{ca}\,\overline{ab})$$
$$= (ab)^2 + \hat\rho^2 - 2\hat\rho\,|(ab)|\cos(\overline{ab}\,\rho).$$

Secondly, suppose $\hat\rho$ negative, then

$$|(ac)| = -\hat\rho,\ \overline{ac} = \bar\rho.$$

In this case also

$$(bc)^2 = (ab)^2 + \hat{\rho}^2 - 2\hat{\rho}\,|(ab)|\cos(\overline{ab}\,\rho).$$

Hence $$(a_{\acute\rho}b)^2 = (ab)^2 - 2\,|(ab)|\,\hat{\rho}\cos(\overline{ab}\,\rho) + \hat{\rho}^2,$$

$$\begin{aligned}\therefore\ (a_{\acute\rho\acute\sigma}b)^2 &= (a_{\acute\rho}b)^2 - 2\,|(a_{\acute\rho}b)|\,\hat{\sigma}\cos(\overline{a_\rho b}\,\sigma) + \hat{\sigma}^2\\
&= (ab)^2 - 2\,|(ab)|\,\hat{\rho}\cos(\overline{ab}\,\rho) + \hat{\rho}^2 + \hat{\sigma}^2\\
&\qquad - 2\hat{\sigma}\,[(a_{\acute\rho}\sigma_{\frac{\pi}{2}}) - (b\sigma_{\frac{\pi}{2}})]\\
&= (ab)^2 - 2\,|(ab)|\,\hat{\rho}\cos(\overline{ab}\,\rho) + \hat{\rho}^2 + \hat{\sigma}^2\\
&\qquad - 2\hat{\sigma}\,[(a\sigma_{\frac{\pi}{2}}) - \hat{\rho}\cos(\rho\sigma) - (b\sigma_{\frac{\pi}{2}})],\ \text{by § 25},\\
&= (ab)^2 - 2\hat{\rho}\,|(ab)|\cos(\overline{ab}\,\rho) - 2\hat{\sigma}\,|(ab)|\cos(\overline{ab}\,\sigma)\\
&\qquad + \hat{\rho}^2 + \hat{\sigma}^2 + 2\hat{\rho}\hat{\sigma}\cos(\rho\sigma).\end{aligned}$$

And it is easy to see that

$$(a_{\acute\rho\acute\sigma}\,._{\acute\omega}b)^2 = (ab)^2 - 2\,|(ab)|\,\Sigma\hat{\rho}\cos(\overline{ab}\,\rho) + \Sigma\hat{\rho}^2 + 2\Sigma\hat{\rho}\hat{\sigma}\cos(\rho\sigma).$$

§ 25. *To express $(a_{\acute\rho\acute\sigma}\ _{\acute\omega}\beta)$ in terms of measures of two elements and vectorial magnitudes.*

Let $a_{\acute\rho} = c$, and consider firstly $\hat{\rho}$ positive, then $|(ac)| = \hat{\rho}$, $\overline{ac} = \rho$.

Now $$\sin(\overline{ac}\,\beta) = \frac{(a\beta) - (c\beta)}{|(ac)|},$$

$$\therefore\ (c\beta) = (a\beta) - \hat{\rho}\sin(\rho\beta)$$

If $\hat{\rho}$ be negative, $\quad |(ac)| = -\hat{\rho},\ \overline{ac} = \bar{\rho}$,

and again $\quad (c\beta) = (a\beta) - \hat{\rho}\sin(\rho\beta)$

Hence in both cases

$$\begin{aligned}(a_\rho\beta) &= (a\beta) - \hat{\rho}\sin(\rho\beta),\\
\therefore\ (a_{\acute\rho\acute\sigma}\beta) &= (a_{\acute\rho}\beta) - \hat{\sigma}\sin(\sigma\beta)\\
&= (a\beta) - \hat{\rho}\sin(\rho\beta) - \hat{\sigma}\sin(\sigma\beta).\end{aligned}$$

It is easy to see that

$$(a_{\acute\rho\acute\sigma}\ _{\acute\omega}\beta) = (a\beta) - \Sigma\hat{\rho}\sin(\rho\beta).$$

§ 26. The general vectorial line is $a_{\acute\rho\acute\sigma\ldots\acute\phi\omega}$, where $\acute\rho, \acute\sigma \ldots \acute\phi$ are vectors and ω a direction. On reaching the point given by the series of vectors $\acute\rho, \acute\sigma \ldots \acute\phi$ we take a line through this point parallel to the given direction.

To find the reduction of the measure $(a_{\hat{\rho}\hat{\sigma}}\ldots{}_{\hat{\phi}_\omega}b)$.

Let the point $a_{\hat{\rho}\hat{\sigma}}\ldots{}_{\hat{\phi}}$ be c; we need $(c_\omega b)$.

$$\begin{aligned}(c_\omega b) = (bc_\omega) &= (bc_\omega) - (cc_\omega)\\ &= (bcc_\omega) = |(bc)|\sin(\overline{bc}c_\omega)\\ &= |(bc)|\sin(\overline{bc}\omega)\\ &= (bc\omega)\\ &= (b\omega) - (c\omega)\\ &= (b\omega) - (a_{\hat{\rho}\hat{\sigma}}\ \ {}_{\hat{\phi}}\,\omega)\\ &= (b\omega) - \{(a\omega) - \Sigma\hat{\rho}\sin(\rho\omega)\}\\ &= (ba\omega) + \Sigma\hat{\rho}\sin(\rho\omega),\end{aligned}$$

$$\therefore\ (a_{\hat{\rho}\hat{\sigma}}\ldots{}_{\hat{\phi}_\omega}b) = (ba\omega) + \Sigma\hat{\rho}\sin(\rho\omega).$$

§ 27. *To reduce the measure* $(a_{\hat{\rho}\hat{\sigma}}\ \ {}_{\hat{\phi}_\omega}\beta)$.

Evidently $$(a_{\hat{\rho}\hat{\sigma}}\ \ {}_{\hat{\phi}_\omega}\beta) = (\omega\beta).$$

§ 28. *Examples.*

1. Shew that $(x_a\beta\gamma) = -(x\beta)\sin(a\gamma) + (x\gamma)\sin(a\beta)$.

We have $(x_a\beta\gamma) = (xx_a)\sin(\beta\gamma) + (x\beta)\sin(\gamma x_a) + (x\gamma)\sin(x_a\beta)$.

2. Prove that the perpendiculars of a triangle intersect.

Let $\left.\begin{matrix}abc\\ \alpha\beta\gamma\end{matrix}\right\}$ be the points and sides of the triangle.

Denote the perpendiculars by $\lambda\mu\nu$.

$$\begin{aligned}(\lambda\mu\nu) &= (a_{\alpha_{\frac{\pi}{2}}}\mu\nu) = (a\mu)\cos(\alpha\nu) - (a\nu)\cos(\alpha\mu)\\ &= -(ab_{\beta_{\frac{\pi}{2}}})\sin(\alpha\gamma) + (ac_{\gamma_{\frac{\pi}{2}}})\sin(\alpha\beta)\\ &= (ab\beta_{\frac{\pi}{2}})\sin(\gamma\alpha) + (ac\gamma_{\frac{\pi}{2}})\sin(\alpha\beta)\\ &= |(ab)|\cos(\gamma\beta)\sin(\gamma\alpha) - |(ac)|\cos(\beta\gamma)\sin(\alpha\beta)\\ &= 0.\end{aligned}$$

3. Let $\left.\begin{matrix}abc\\ \alpha\beta\gamma\end{matrix}\right\}$, $\left.\begin{matrix}a'b'c'\\ \alpha'\beta'\gamma'\end{matrix}\right\}$ be two triangles: lines are drawn through a, b, c perpendicular to α', β', γ', forming the triangle whose sides are λ_1, μ_1, ν_1: similarly through a', b', c' are drawn lines perpendicular to α, β, γ, forming the triangle λ_2, μ_2, ν_2.

Shew that $\dfrac{(\lambda_1\mu_1\nu_1)}{(\lambda_2\mu_2\nu_2)} = -\dfrac{R}{R'}$, where R, R' are the circum-radii of the triangles abc, $a'b'c'$.

$$\begin{aligned}(\lambda_1\mu_1\nu_1) &= (a_{\alpha'_{\frac{\pi}{2}}}\mu_1\nu_1) = (a\mu_1)\cos(\alpha'\nu_1) - (a\nu_1)\cos(\alpha'\mu_1)\\ &= |(ab)|\cos(\overline{ab}\beta')\sin(\gamma\alpha') + |(ac)|\cos(\overline{ac}\gamma')\sin(\alpha'\beta')\\ &= 2R\{\sin(\alpha\beta)\cos(\beta'\gamma)\sin(\gamma'\alpha') - \sin(\gamma\alpha)\cos(\beta\gamma')\sin(\alpha'\beta')\}.\end{aligned}$$

The expression within the brackets is symmetrical in a, a'; β, β'; γ, γ' but for sign.

Hence the result.

Corollary. If the perpendiculars from abc on to $a'\beta'\gamma'$ be concurrent, the perpendiculars from $a'b'c'$ on $a\beta\gamma$ are also concurrent.

4. Reduce the measure $(a_{\acute\rho} b\gamma)$.

$$\begin{aligned}(a_{\acute\rho} b\gamma) &= (a_{\acute\rho}\gamma) - (b\gamma)\\ &= (a\gamma) - \hat\rho \sin(\rho\gamma) - (b\gamma)\\ &= (ab\gamma) - \hat\rho \sin(\rho\gamma).\end{aligned}$$

5. Reduce the measure $(a_{\acute\rho} b_{\acute\sigma} \gamma)$.

$$\begin{aligned}(a_{\acute\rho} b_{\acute\sigma} \gamma) &= (a_{\acute\rho}\gamma) - (b_{\acute\sigma}\gamma)\\ &= (a\gamma) - (b\gamma) - \hat\rho \sin(\rho\gamma) + \hat\sigma \sin(\sigma\gamma)\\ &= (ab\gamma) - \hat\rho \sin(\rho\gamma) + \hat\sigma \sin(\sigma\gamma).\end{aligned}$$

6. Reduce $(a_{\acute\rho} bc)$.

$$(a_{\acute\rho} bc) = (abc) + \hat\rho\,(bc\rho).$$

7. Reduce $(a_{\acute\rho} b_{\acute\sigma} c)$.

$$\begin{aligned}(a_{\acute\rho} b_{\acute\sigma} c) &= (ab_{\acute\sigma} c) + \hat\rho\,(b_{\acute\sigma} c\rho)\\ &= (b_{\acute\sigma} ca) + \hat\rho\,[(b\rho) - \hat\sigma \sin(\sigma\rho) - (c\rho)]\\ &= (abc) + \hat\rho\,(bc\rho) + \hat\sigma\,(ca\sigma) + \hat\rho\hat\sigma \sin(\rho\sigma).\end{aligned}$$

8. Reduce $(a_{\acute\lambda} b_{\acute\mu} c_{\acute\nu})$.

$$\begin{aligned}(a_{\acute\lambda} b_{\acute\mu} c_{\acute\nu}) &= (abc_{\acute\nu}) + \hat\lambda\,(bc_{\acute\nu}\lambda) + \hat\mu\,(c_{\acute\nu} a\mu) + \hat\lambda\hat\mu \sin(\lambda\mu)\\ &= (abc) + \Sigma\hat\lambda\,(bc\lambda) + \Sigma\hat\mu\hat\nu \sin(\mu\nu).\end{aligned}$$

9. Reduce $(a_{\acute\lambda_1 \acute\lambda_2 \dots \acute\lambda_n} b_{\acute\mu_1 \acute\mu_2 \dots \acute\mu_n} c_{\acute\nu_1 \acute\nu_2 \dots \acute\nu_n})$.

$$\begin{aligned}&(a_{\acute\lambda_1 \acute\lambda_2 \dots \acute\lambda_n} b_{\acute\mu_1 \acute\mu_2 \dots \acute\mu_n} c_{\acute\nu_1 \acute\nu_2 \dots \acute\nu_n})\\ &= (a_{\acute\lambda_1 \acute\lambda_2 \dots \acute\lambda_{n-1}} b_{\acute\mu_1 \acute\mu_2 \dots \acute\mu_{n-1}} c_{\acute\nu_1 \acute\nu_2 \dots \acute\nu_{n-1}})\\ &\quad + \lambda_n\,(b_{\acute\mu_1 \acute\mu_2 \dots \acute\mu_{n-1}} c_{\nu_1 \acute\nu_2 \dots \acute\nu_{n-1}} \lambda_n)\\ &\quad + \mu_n\,(c_{\acute\nu_1 \acute\nu_2 \dots \acute\nu_{n-1}} a_{\acute\lambda_1 \acute\lambda_2 \dots \acute\lambda_{n-1}} \mu_n) + \nu_n\,(a_{\acute\lambda_1 \acute\lambda_2 \dots \acute\lambda_{n-1}} b_{\mu_1 \mu_2 \dots \acute\mu_{n-1}} \nu_n)\\ &\quad + \hat\mu_n \hat\nu_n \sin(\mu_n\nu_n) + \hat\nu_n \hat\lambda_n \sin(\nu_n\lambda_n) + \hat\lambda_n \hat\mu_n \sin(\lambda_n\mu_n)\\ &= (a_{\acute\lambda_1 \acute\lambda_2 \dots \acute\lambda_{n-1}} b_{\acute\mu_1 \acute\mu_2 \dots \acute\mu_{n-1}} c_{\acute\nu_1 \acute\nu_2 \dots \acute\nu_{n-1}})\\ &\quad + \hat\lambda_n\,(bc\lambda_n) + \hat\mu_n\,(ca\mu_n) + \hat\nu_n\,(ab\nu_n)\\ &\quad + \hat\mu_n \hat\nu_n \sin(\mu_n\nu_n) + \hat\nu_n \hat\lambda_n \sin(\nu_n\lambda_n) + \hat\lambda_n \hat\mu_n \sin(\lambda_n\mu_n)\\ &\quad + \hat\lambda_n \sum_1^{n-1} \hat\mu_r \sin(\lambda_n\mu_r) + \hat\lambda_n \sum_1^{n-1} \hat\nu_r \sin(\nu_r\lambda_n)\\ &\quad + \hat\mu_n \sum_1^{n-1} \hat\nu_r \sin(\mu_n\nu_r) + \hat\mu_n \sum_1^{n-1} \hat\lambda_r \sin(\lambda_r\mu_n)\\ &\quad + \hat\nu_n \sum_1^{n-1} \hat\lambda_r \sin(\nu_n\lambda_r) + \hat\nu_n \sum_1^{n-1} \hat\mu_r \sin(\mu_r\nu_n),\end{aligned}$$

$$\begin{aligned}\therefore\ &(a_{\acute\lambda_1 \acute\lambda_2 \dots \acute\lambda_n} b_{\acute\mu_1 \acute\mu_2 \dots \acute\mu_n} c_{\acute\nu_1 \acute\nu_2 \dots \acute\nu_n})\\ &= (abc) + \sum_{\lambda,\mu,\nu} \sum_{r=1}^{n} (bc\lambda_r)\\ &\quad + \sum_{\lambda,\mu,\nu} \sum_r \sum_s \sin(\mu_r\nu_s)\,\hat\mu_r \hat\nu_s.\end{aligned}$$

10. Shew that $(a_{\acute{\rho}\sigma}\beta\gamma) = (a_\sigma\beta\gamma) + \hat{\rho}\sin(\beta\gamma)\sin(\rho\sigma)$.

We have
$$\begin{aligned}(a_{\acute{\rho}\sigma}\beta\gamma) &= -(a_{\acute{\rho}}\beta)\sin(\sigma\gamma) + (a_{\acute{\rho}}\gamma)\sin(\sigma\beta)\\ &= -\{(a\beta) - \hat{\rho}\sin(\rho\beta)\}\sin(\sigma\gamma)\\ &\quad + \{(a\gamma) - \hat{\rho}\sin(\rho\gamma)\}\sin(\sigma\beta)\\ &= -(a\beta)\sin(\sigma\gamma) + (a\gamma)\sin(\sigma\beta)\\ &\quad + \hat{\rho}\sin(\rho\beta)\sin(\sigma\gamma) - \hat{\rho}\sin(\rho\gamma)\sin(\sigma\beta)\\ &= (a_\sigma\beta\gamma) + \hat{\rho}\sin(\beta\gamma)\sin(\rho\sigma).\end{aligned}$$

11. Shew that $(x_{\acute{a}\lambda}y_{\acute{\beta}\mu}z_{\acute{\gamma}\nu}) = (x_\lambda y_\mu z_\nu) + \Sigma\hat{a}\sin(\mu\nu)\sin(a\lambda)$.

$$\begin{aligned}(x_{\acute{a}\lambda}y_{\acute{\beta}\mu}z_{\acute{\gamma}\nu}) &= (x_\lambda y_{\acute{\beta}\mu}z_{\acute{\gamma}\nu}) + \hat{a}\sin(\mu\nu)\sin(a\lambda)\\ &= (y_\mu z_{\acute{\gamma}\nu}x_\lambda) + \hat{a}\sin(\mu\nu)\sin(a\lambda)\\ &\quad + \hat{\beta}\sin(\nu\lambda)\sin(\beta\mu)\\ &= (z_\nu x_\lambda y_\mu) + \hat{a}\sin(\mu\nu)\sin(a\lambda)\\ &\quad + \hat{\beta}\sin(\nu\lambda)\sin(\beta\mu) + \hat{\gamma}\sin(\lambda\mu)\sin(\gamma\nu).\end{aligned}$$

12. Shew that $(x_{\acute{a}_1\acute{a}_2\ldots\acute{a}_n\lambda}y_{\acute{\beta}_1\acute{\beta}_2\ldots\acute{\beta}_n\mu}z_{\acute{\gamma}_1\acute{\gamma}_2\ldots\acute{\gamma}_n\nu})$
$$= (x_\lambda y_\mu z_\nu) + \sin(\mu\nu)\sum_1^n \hat{a}_r\sin(a_r\lambda) + \sin(\nu\lambda)\sum_1^n \hat{\beta}_r\sin(\beta_r\mu)$$
$$+ \sin(\lambda\mu)\sum_1^n \hat{\gamma}_r\sin(\gamma_r\nu).$$

13. If a, β, γ, δ be four lines, shew that
$$\Sigma(\beta\gamma\delta)\sin(a\delta) = 0.$$
$$\Sigma(\beta\gamma\delta)\sin(a\delta) = \Sigma\{(o\beta)\sin(\gamma\delta) + (o\gamma)\sin(\delta\beta) + (o\delta)\sin(\beta\gamma)\}\sin(a\delta) = 0$$

14. Let $\begin{matrix}abc\\ a\beta\gamma\end{matrix}\Big\}$ be a triangle and λ any line. p, q, r are the feet of the perpendiculars from a, b, c on λ: shew that the perpendiculars from p, q, r on a, β, γ are concurrent.

The perpendicular from p on a is represented by $a_{\lambda_{\frac{\pi}{2}},\,-(a\lambda)\,a_{\frac{\pi}{2}}}$.

And we have
$$\begin{aligned}&(a_{\lambda_{\frac{\pi}{2}},\,-(a\lambda),\,a_{\frac{\pi}{2}}}\; b_{\lambda_{\frac{\pi}{2}},\,-(b\lambda),\,\beta_{\frac{\pi}{2}}}\; c_{\lambda_{\frac{\pi}{2}},\,-(c\lambda),\,\gamma_{\frac{\pi}{2}}})\\ &= (a_{a_{\frac{\pi}{2}}}b_{\beta_{\frac{\pi}{2}}}c_{\gamma_{\frac{\pi}{2}}}) - \Sigma(a\lambda)\sin(\beta\gamma)\sin(a\lambda)\\ &= 0 - \Sigma(\beta\gamma\lambda)\sin(a\lambda) = 0.\end{aligned}$$

The point of intersection of the perpendiculars has been called the *orthopole* of λ in regard to the triangle.

15. If along the perpendiculars from a, b, c on any line λ, we measure off distances equal to the perpendiculars from the angular points of the medial triangle on this line, determining the points p, q, r, shew that the perpendiculars from p, q, r on a, β, γ are concurrent.

16. Let $\begin{matrix}abc\\ a\beta\gamma\end{matrix}\Big\}$, $\begin{matrix}a'b'c'\\ a'\beta'\gamma'\end{matrix}\Big\}$ be two triangles. Let p, q, r be the feet of the perpendiculars from a, b, c on any line δ; p', q', r' those from a', b', c' on δ.

Let the perpendiculars from p, q, r on a', β', γ' be λ_1, μ_1, ν_1, and the perpendiculars from p', q', r' on a, β, γ be λ_2, μ_2, ν_2. Shew that $\dfrac{(\lambda_1\mu_1\nu_1)}{(\lambda_2\mu_2\nu_2)}=-\dfrac{R}{R'}$.

Now $$(\lambda_1\mu_1\nu_1)=(a_{\lambda_{\frac{\pi}{2}},\,-(a\lambda),\,a'_{\frac{\pi}{2}}}\ \ b_{\lambda_{\frac{\pi}{2}},\,-(b\lambda),\,\beta'_{\frac{\pi}{2}}}\ \ c_{\lambda_{\frac{\pi}{2}},\,-(c\lambda),\,\gamma'_{\frac{\pi}{2}}})$$
$$=(a_{a'_{\frac{\pi}{2}}}\ b_{\beta'_{\frac{\pi}{2}}}\ c_{\gamma'_{\frac{\pi}{2}}})+\Sigma\,(a\lambda)\sin(\beta'\gamma')\sin(a'\lambda)$$
$$=A\,.\,R+\frac{1}{2R'}\Sigma\,(a\lambda)\,(b'c'\lambda),$$ A being symmetrical in regard to the triangles, but for sign, from Ex. 3

$=R\times$ function symmetrical in the elements of the two triangles, but for sign. Hence the result.

17. The circum-centre of a triangle is represented by s. Find the area, i.e. half of the standard measure, of the pedal triangle of the point $s_{\hat{\rho}}$.

The circum-centre is the point where
$$(sa)=R\cos(\beta\gamma),\quad(s\beta)=R\cos(\gamma a),\quad(s\gamma)=R\cos(a\beta).$$

Then if p, q, r be the feet of the perpendiculars from a point m on a, β, γ,
$$(pqr)=(m_{a_{\frac{\pi}{2}},\,-(ma)}\ \ m_{\beta_{\frac{\pi}{2}},\,-(m\beta)}\ \ m_{\gamma_{\frac{\pi}{2}},\,-(m\gamma)})$$
$$=\Sigma\,(m\beta)\,(m\gamma)\sin(\beta\gamma).$$

Then if $m=s_{\hat{\rho}}$,
$$(pqr)=(s_{\hat{\rho}}\beta)\,(s_{\rho}\gamma)\sin(\beta\gamma)+\ldots+\ldots$$
$$=\{(s\beta)-\hat{\rho}\sin(\rho\beta)\}\,\{(s\gamma)-\hat{\rho}\sin(\rho\gamma)\}\sin(\beta\gamma)+\ldots+\ldots$$
$$=\{R\cos(\gamma a)-\hat{\rho}\sin(\rho\beta)\}\,\{R\cos(a\beta)-\hat{\rho}\sin(\rho\gamma)\}\sin(\beta\gamma)+\ldots+\ldots$$
$$=R^2\,\Sigma\cos(\gamma a)\cos(a\beta)\sin(\beta\gamma)+R\hat{\rho}\,\Sigma\sin(\rho a)\sin(\beta\gamma)$$
$$+\hat{\rho}^2\,\Sigma\sin(\rho\beta)\sin(\rho\gamma)\sin(\beta\gamma)$$
$$=(R^2-\hat{\rho}^2)\sin(\beta\gamma)\sin(\gamma a)\sin(a\beta).$$

18. Shew that
$$\sum_{a,\,b,\,c}(o_{\hat{\rho}}a)^2\,(pbc)=\Sigma\,(oa)^2\,(pbc)+(abc)\,\hat{\rho}^2+2\,(abc)\,\hat{\rho}\,|(po)|\cos(\overline{p}o\rho).$$
$$\Sigma\,(o_{\rho}a)^2\,(pbc)=\Sigma\,\{(oa)^2-2\hat{\rho}\,|(oa)|\cos(\overline{oa}\rho)+\hat{\rho}^2\}\,(pbc)$$
$$=\Sigma\,(oa)^2\,(pbc)+\hat{\rho}^2\,(abc)-2\hat{\rho}\,\{(o\rho_{\frac{\pi}{2}})-(a\rho_{\frac{\pi}{2}})\}\,(pbc)$$
$$=\Sigma\,(oa)^2\,(pbc)+\hat{\rho}^2\,(abc)-2\hat{\rho}\,(o\rho_{\frac{\pi}{2}})\,(abc)+2\hat{\rho}\,(p\rho_{\frac{\pi}{2}})\,(abc)$$
$$=\Sigma\,(oa)^2\,(pbc)+\hat{\rho}^2\,(abc)+2\hat{\rho}\,(abc)\,|(po)|\cos(\overline{po}\rho).$$

19. Hence shew that
$$\sum_{a,\,b,\,c}(oa)^2\,(pbc)=\{R^2+(po)^2-(ps)^2\}\,(abc),$$
where s is the circum-centre of a, b, c and R the circum-radius
$$\sum_{a,\,b,\,c}(oa)^2\,(pbc)=\Sigma\,(s_{\omega,\,\rho}a)^2\,(pbc),\quad\text{where }\omega=\overline{so},\ \rho=|(so)|$$
$$=\Sigma\,(sa)^2\,(pbc)+\rho^2\,(abc)+2\rho\,(abc)\,|(ps)|\cos(\overline{ps}\omega)$$
$$=\{R^2+\rho^2+2\rho\,|(ps)|\cos(\overline{ps}\,\overline{so})\}\,(abc)$$
$$=\{R^2+(po)^2-(ps)^2\}\,(abc).$$

CHAPTER III

REDUCTION OF MEASURES CONTAINING EQUATIONAL ELEMENTS

§ 29. It is evident that an equation

$$f\{(x\alpha_1),\ (x\alpha_2) \ldots |(xa_1)|,\ |(xa_2)| \ldots\} = 0,$$

where α_1, $\alpha_2 \ldots a_1$, $a_2 \ldots$ are fixed, is a locus of x.

We shall consider the following locus a linear function of $(x\alpha_1)$, $(x\alpha_2) \ldots$ namely,

$$\Sigma a_r (x\alpha_r) + a = 0.$$

Let y, z be two points on the locus, then

$$\Sigma a_r (y\alpha_r) + a = 0,$$
$$\Sigma a_r (z\alpha_r) + a = 0.$$

Hence by subtracting

$$\Sigma a_r (yz\alpha_r) = 0,$$

i.e.

$$\Sigma a_r \sin (\overline{yz}\alpha_r) = 0,$$

which equation determines the direction of $\overline{yz}$.

Hence the locus must be a straight line.

Conversely, it may be shewn that any line can be expressed in the form of a linear equation. For let ξ be the line, and let α, β be any two lines, concurrent with ξ.

Let x be any point on ξ, then

$$(x\beta) \sin (\xi\alpha) + (x\alpha) \sin (\beta\xi) = 0,$$

and by taking any two lines γ, δ concurrent with α, we have

$$(x\alpha) \sin (\gamma\delta) + (x\gamma) \sin (\delta\alpha) + (x\delta) \sin (\alpha\gamma) = 0.$$

Hence β, γ, δ being any three arbitrary lines the equation of ξ may be expressed as a linear function of $(x\beta)$, $(x\gamma)$, $(x\delta)$.

It is important to notice that the locus given by a linear equation is according to our stipulation two lines; namely, a line and the line with same position but reversed direction. The signs of the square roots occurring in the following are therefore necessarily indeterminate.

§ 30. *To reduce the measure* $(\Sigma a_r (x\alpha_r) + a = 0\beta)$.

If y, z be two points on $\xi \equiv \Sigma a_r (x\alpha_r) + a = 0$ we have seen that

$$\Sigma a_r \sin (\overline{yz}\alpha_r) = 0,$$

i.e.
$$\Sigma a_r \sin (\xi\alpha_r) = 0.$$

$$\therefore\ \Sigma a_r \sin \{(\xi\beta) - (\alpha_r\beta)\} = 0.$$

$$\therefore\ \sin (\xi\beta)\, \Sigma a_r \cos (\alpha_r \beta) = \cos (\xi\beta)\, \Sigma a_r \sin (\alpha_r \beta),$$

$$\therefore\ \tan (\xi\beta) = \frac{\Sigma a_r \sin (\alpha_r\beta)}{\Sigma a_r \cos (\alpha_r\beta)}.$$

Hence
$$\sin (\xi\beta) = \frac{\Sigma a_r \sin (\alpha_r\beta)}{\sqrt{\Sigma a_r^2 + 2\Sigma a_r a_s \cos (\alpha_r\alpha_s)}},$$

$$\cos (\xi\beta) = \frac{\Sigma a_r \cos (\alpha_r\beta)}{\sqrt{\Sigma a_r^2 + 2\Sigma a_r a_s \cos (\alpha_r\alpha_s)}}.$$

Let us give ξ a certain sense. With this sense we have

$$\sin (\xi\beta) = \frac{\Sigma a_r \sin (\alpha_r\beta)}{m \left| \sqrt{\Sigma a_r^2 + 2\Sigma a_r a_s \cos (\alpha_r\alpha_s)} \right|} \quad \text{.........(i)},$$

where m is either $+1$ or -1.

Then
$$\cos (\xi\beta) = \frac{\Sigma a_r \cos (\alpha_r\beta)}{m \left| \sqrt{\Sigma a_r^2 + 2\Sigma a_r a_s \cos (\alpha_r\alpha_s)} \right|} \quad \text{.........(ii)},$$

since the sign of the tangent is independent of the sense of ξ.

Suppose γ any other line.

Then

$$\sin (\xi\gamma) = \sin \{(\xi\beta) + (\beta\gamma)\} = \sin (\xi\beta) \cos (\beta\gamma) + \cos (\xi\gamma) \sin (\beta\gamma)$$
$$= \frac{\Sigma a_r \sin (\alpha_r\gamma)}{m \left| \sqrt{\Sigma a_r^2 + 2\Sigma a_r a_s \cos (\alpha_r\alpha_s)} \right|},$$

substituting from (i) and (ii).

Hence the sign of the square root depends only on the particular sense of ξ chosen.

§ 31. *To reduce the measure* $(\Sigma a_r (x\alpha_r) + a = 0\, b)$.

Let y be any point on the line, and let ω denote its direction.

Then

$$
\begin{aligned}
(\Sigma a_r (x\alpha_r) + a = 0b) &= (y_\omega b) \\
&= (by\omega) \\
&= |(yb)| \sin (\omega \overline{yb}) \\
&= |(yb)| \frac{\Sigma a_r \sin (\alpha_r \overline{yb})}{\sqrt{\Sigma a_r^2 + 2\Sigma a_r a_s \cos (\alpha_r \alpha_s)}} \\
&= \frac{\Sigma a_r (by\alpha_r)}{\sqrt{\Sigma a_r^2 + 2\Sigma a_r a_s \cos (\alpha_r \alpha_s)}} \\
&= \frac{\Sigma a_r (b\alpha_r) - \Sigma a_r (y\alpha_r)}{\sqrt{\Sigma a_r^2 + 2\Sigma a_r a_s \cos (\alpha_r \alpha_s)}}.
\end{aligned}
$$

$$
\therefore \ (\Sigma a_r (x\alpha_r) + a = 0b) = \frac{\Sigma a_r (b\alpha_r) + a}{\sqrt{\Sigma a_r^2 + 2\Sigma a_r a_s \cos (\alpha_r \alpha_s)}}.
$$

Supposing the line $\Sigma a_r (x\alpha_r) + a = 0$ to have a specified sense, it is important to notice that the square roots occurring in this and the former section have the same sign.

§ 32. Next we shall shew that

$$\Sigma A_r (\xi a_r) + \Sigma B_r \cos (\xi \beta_r) = 0,$$

where $a_1, a_2 \ldots, \beta_1, \beta_2 \ldots$ are fixed, is the equation of a point.

A given line may be iepresented by $c_{\omega, r, \phi}$ where c is an arbitrary point. Let this satisfy the equation, then

$$\Sigma A_r (c_{\omega, r, \phi} a_r) + \Sigma B_r \cos (c_{\omega, r, \phi} \beta_r) = 0,$$

$$\therefore \ \Sigma A_r \{(a_r c\phi) + r \sin (\omega\phi)\} + \Sigma B_r \cos (\phi\beta_r) = 0,$$

$$\therefore \ r \sin (\omega\phi) \Sigma A_r = - \Sigma A_r (a_r c\phi) - \Sigma B_r \cos (\phi\beta_r).$$

We may change ϕ to $\phi_{\frac{\pi}{2}}$ and we have

$$r \cos (\omega\phi) \Sigma A_r = - \Sigma A_r (a_r c\phi_{\frac{\pi}{2}}) - \Sigma B_r \sin (\phi\beta_r).$$

Squaring and adding

$$
\begin{aligned}
r^2 (\Sigma A_r)^2 = \Sigma A_r^2 (a_r c)^2 &+ 2 \sum_{r \neq s} A_r A_s |(a_r c)(a_s c)| \cos (\overline{a_r c}\, \overline{a_s c}) \\
&+ 2\Sigma A_r B_s |(a_r c)| \sin (\overline{a_r c}\, \beta_s) \\
&+ \Sigma B_r^2 + 2\Sigma B_r B_s \cos (\beta_r \beta_s).
\end{aligned}
$$

Since the right-hand side is independent of ω, ϕ all the lines must pass through the same point. In other words, the equation is that of a point.

Conversely, any point can be represented by an equation of the form considered. For let o be the point, take two points a, b collinear with o. Let ξ be any line incident in o.

Then $$(a\xi)(bo) = (b\xi)(ao).$$

Then taking any two lines β, γ we have from b, ξ, β, γ

$$(b\xi)\sin(\beta\gamma) - (b\beta)\sin(\xi\gamma) + (b\gamma)\sin(\xi\beta) = (\xi\beta\gamma).$$

Hence

$$(a\xi)\sin(\beta\gamma)(bo) = (ao)\left[-(b\beta)\cos(\xi\gamma_{\frac{\pi}{2}}) + (b\gamma)\cos(\xi\beta_{\frac{\pi}{2}}) + (\xi\beta\gamma)\right].$$

a, β, γ are any three elements, and the equation is of the form considered, proving the theorem.

§ 33. *To reduce the measure*

$$(\Sigma A_r(\xi a_r) + \Sigma B_r \cos(\xi\beta_r) = 0c)^2.$$

We have seen that

$$\begin{aligned}(\Sigma A_r(\xi a_r) + \Sigma B_r\cos(\xi\beta_r) = 0c)^2(\Sigma A_r)^2 \\ = \Sigma A_r^2(a_rc)^2 + 2\sum_{r\neq s} A_rA_s\,|(a_rc)(a_sc)|\cos(\overline{a_rc}\,\overline{a_sc}) \\ + 2\Sigma A_rB_s(a_rc\beta_s) + \Sigma B_r^2 + 2\Sigma B_rB_s\cos(\beta_r\beta_s),\end{aligned}$$

which is the required reduction.

§ 34. *To reduce* $(\Sigma A_r(\xi a_r) + \Sigma B_r\cos(\xi\beta_r) = 0\,\gamma)$.

Let c, d be two points on γ, and let a be the point

$$\Sigma A_r(\xi a_r) + \Sigma B_r\cos(\xi\beta_r) = 0.$$

Then $\overline{ac}$, $\overline{ad}$ will satisfy the equation.

$$\therefore\ \Sigma A_r(\overline{ac}\,a_r) + \Sigma B_r\cos(\overline{ac}\,\beta_r) = 0,$$

$$\therefore\ \Sigma A_r(aca_r) + \Sigma B_r(ac\beta_{r\frac{\pi}{2}}) = 0.$$

Similarly $$\Sigma A_r(ada_r) + \Sigma B_r(ad\beta_{r\frac{\pi}{2}}) = 0.$$

$\therefore$ subtracting $$\Sigma A_r\,|(a_ra)|\,(cd\overline{a_ra}) - \Sigma B_r(cd\beta_{r\frac{\pi}{2}}) = 0,$$

$$\therefore\ \Sigma A_r\,|(a_ra)|\sin(\gamma\overline{a_ra}) - \Sigma B_r\cos(\gamma\beta_r) = 0,$$

i.e. $$\Sigma A_r(aa_r\gamma) - \Sigma B_r\cos(\gamma\beta_r) = 0,$$

$$\therefore\ (a\gamma)\Sigma A_r = \Sigma A_r(a_r\gamma) + \Sigma B_r\cos(\gamma\beta_r).$$

Hence $$\begin{aligned}(\Sigma A_r(\xi a_r) + \Sigma B_r\cos(\xi\beta_r) = 0\,\gamma)\,\Sigma A_r \\ = \Sigma A_r(\gamma a_r) + \Sigma B_r\cos(\gamma\beta_r).\end{aligned}$$

§ 35. In the case in which $\Sigma A_r = 0$ the foregoing results break down. We shall consider this case.

From the equation

$$\Sigma A_r (\xi a_r) + \Sigma B_r \cos(\xi\beta_r) = 0,$$

subtract $\Sigma A_r (\xi c)$, where c is an arbitrary point. We get

$$\Sigma A_r |(a_r c)| \sin(\overline{a_r c}\xi) + \Sigma B_r \cos(\beta_r \xi) = 0.$$

Let γ be any line, and let $(\xi\gamma) = \theta$.

Then $\Sigma A_r |(a_r c)| \sin\{(a_r c\gamma) - \theta\} + \Sigma B_r \cos\{(\beta_r\gamma) - \theta\} = 0.$

Hence $\quad \Sigma A_r \{(a_r c\gamma)\cos\theta - \sin\theta\,(a_r c\gamma_{\frac{\pi}{2}})\}$

$$+ \Sigma B_r \{\cos(\beta_r\gamma)\cos\theta + \sin(\beta_r\gamma)\sin\theta\} = 0.$$

Hence $\quad \{\Sigma A_r (a_r\gamma) + \Sigma B_r \cos(\beta_r\gamma)\}\cos\theta$

$$= \{\Sigma A_r (a_r\gamma_{\frac{\pi}{2}}) - \Sigma B_r \sin(\beta_r\gamma)\}\sin\theta,$$

giving θ independent of the particular line chosen. Hence the equation represents a direction.

§ 36. *To reduce* $(\Sigma A_r(\xi a_r) + \Sigma B_r \cos(\xi\beta_r) = 0\ \gamma)$, *when* $\Sigma A_r = 0$.

We have from § 35

$$\tan(\Sigma A_r (\xi a_r) + \Sigma B_r \cos(\xi B_r) = 0\ \gamma)$$

$$= \frac{\Sigma A_r (a_r\gamma) + \Sigma B_r \cos(\beta_r\gamma)}{\Sigma A_r (a_r\gamma_{\frac{\pi}{2}}) - \Sigma B_r \sin(\beta_r\gamma)}.$$

§ 37. *Examples.*

1. Shew that

$$\tan(\Sigma a_r (x a_r) + a = 0\ \ \Sigma b_r (x\beta_r) + b = 0) = \frac{\sum_r\sum_s a_r b_s \sin(a_r\beta_s)}{\sum_r\sum_s a_r b_s \cos(a_r\beta_s)}.$$

Now $\quad \sin(\Sigma a_r (x a_r) + a = 0\ \beta) = \dfrac{\Sigma a_r \sin(\beta a_r)}{\sqrt{\Sigma a_r^2 + 2\Sigma a_r a_s \cos(a_r a_s)}},$

$$\therefore \sin(\Sigma a_r (x a_r) + a = 0\ \ \Sigma b_r (x\beta_r) + b = 0)$$

$$= \frac{\Sigma a_r \sin(\Sigma b_r (x\beta_r) + b = 0\ a_r)}{\sqrt{\Sigma a_r^2 + 2\Sigma a_r a_s \cos(a_r a_s)}}$$

$$= \frac{\sum_r a_r \{\sum_s b_s \sin(a_r\beta_s)/\sqrt{\Sigma b_r^2 + 2\Sigma b_r b_s \cos(\beta_r\beta_s)}\}}{\sqrt{\Sigma a_r^2 + 2\Sigma a_r a_s \cos(a_r a_s)}}$$

$$= \sum_r\sum_s a_r b_s \sin(a_r\beta_s)/\Omega_a\Omega_b,$$

where $\quad \Omega_a^2 = \Sigma a_r^2 + 2\Sigma a_r a_s \cos(a_r a_s),$

with a similar expression for the cosine.

2. Shew that

$$(\Sigma a_r (xa_r) + a = 0 \;\; \Sigma b_r (x\beta_r) + b = 0 \;\; \Sigma c_r (x\gamma_r) + c = 0) \; \Omega_a \Omega_b \Omega_c$$
$$= \sum_r \sum_s \sum_t a_r b_s c_t (a_r \beta_s \gamma_t)$$
$$+ a \sum_r \sum_s b_r c_s \sin(\beta_r \gamma_s) + b \sum_r \sum_s c_r a_s \sin(\gamma_r a_s)$$
$$+ c \sum_r \sum_s a_r b_s \sin(a_r \beta_s).$$

Let o be any point, then we have

$$(\Sigma a_r (xa_r) + a = 0 \;\; \Sigma b_r (x\beta_r) + b = 0 \;\; \Sigma c_r (x\gamma_r) + c = 0)$$
$$= \sum_{a,b,c} (\Sigma a_r (xa_r) + a = 0 \;\; o) \sin(\Sigma b_r (x\beta_r) + b = 0 \;\; \Sigma c_r (x\gamma_r) + c = 0)$$
$$= \sum_{a,b,c} \{\Sigma a_r (oa_r) + a\} \sum_s \sum_t b_s c_t \sin(\beta_s \gamma_t) / \Omega_a \Omega_b \Omega_c$$
$$= \sum_{a,b,c} \{\sum_r \sum_s \sum_t a_r b_s c_t (oa_r) \sin(\beta_s \gamma_t) + a \sum_s \sum_t b_s c_t \sin(\beta_s \gamma_t)\} / \Omega_a \Omega_b \Omega_c$$
$$= \{\sum_r \sum_s \sum_t a_r b_s c_t (a_r \beta_s \gamma_t) + \sum_{a,b,c} a \sum_r \sum_s b_s c_t \sin(\beta_s \gamma_t)\} / \Omega_a \Omega_b \Omega_c.$$

3. Shew that

$$\sin(\Sigma B_r \cos(\xi\beta_r) = 0 \; \gamma) = \frac{\Sigma B_r \sin(\gamma\beta_r)}{\sqrt{\Sigma B_r^2 + 2\Sigma B_r B_s \cos(\beta_r \beta_s)}}.$$

§ 38. We may now explain fully the general method of procedure. Let there be n elements, whose relative properties are our consideration; also m algebraic quantities occurring in a system of vectors; the directions of the vectors, we shall include in the n elements. Also p algebraic quantities occurring in the coefficients of equational elements. The elements in the equation we shall include in the n elements The fact that the standard measure of three lines is in itself irreducible complicates matters.

If all the n elements be lines, introduce an arbitrary point. This enables us to reduce the measure of three lines. So we shall suppose among the n elements there is at least *one* point.

Then any measure of these elements and elements derived from them in any of the three ways stated in § 3, can be reduced to an algebraic-trigonometric function of the ${}^nc_2$ measures of the n elements, taken two by two together, and the m and p algebraic quantities. We shall only consider such geometry in which elements are derived in one of the three ways stated in § 3 and no more. Then a property amongst the elements is the vanishing of a function among measures of the n elements and derived elements. By reducing the measures to algebraic-trigonometric functions of the ${}^nc_2$ measures of the n elements taken two by two, we have to

prove the vanishing of a function of these ${}^nc_2$ measures, and the m and p algebraic quantities.

Again, let the n elements be composed of n_p points and n_l lines. When $n_p = 1$, there are $n_l - 2$ relations between the measures two by two of the n_l lines and there are no other relations. When $n_p > 1$, there are $\frac{(n-2)(n-3)}{2}$ relations between the measures two by two of the n elements. See § 21.

Our task is then to prove the vanishing of the function of the ${}^nc_2$ measures and m and p quantities by means of these and only these relations between the measures.

DIFFERENTIAL GEOMETRY

CHAPTER IV

DIFFERENTIATION OF MEASURES OF SIMPLE ELEMENTS

§ 39. Let x be any point, and x' a consecutive point. Then $|(xx')|$ we denote by δx, $\overline{xx'}$ by τx.

Again let ξ be any line, and ξ' a consecutive position. $(\xi\xi')$ we denote by $\partial\xi$, $\overline{\xi\xi'}$ by $p\xi$.

§ 40. From the preceding it may be shewn that

$$\underset{\theta \to 0}{\mathrm{L}} \frac{\sin\theta}{\theta} = k, \text{ a constant.}$$

The precise value of k is still at our disposal. We shall suppose then that

$$\underset{\theta \to 0}{\mathrm{L}} \frac{\sin\theta}{\theta} = 1 \quad \text{.....................(XIX).}$$

With this stipulation it may be shewn that the sine and cosine functions may now be expressed as the usual infinite series.

§ 41. *To differentiate* $|(xa)|$.

The point a is supposed fixed. We define the differential coefficient or derivative of $|(xa)|$ as

$$\underset{x \to x'}{\mathrm{L}} \frac{|(x'a)| - |(xa)|}{|(xx')|}.$$

We represent this by $\dfrac{d\,|(xa)|}{dx}$.

Then
$$\frac{d\,|(xa)|}{dx} = \underset{x \to x'}{\mathrm{L}} \frac{|(x'a)| - |(xa)|}{|(xx')|}$$
$$= \underset{x \to x'}{\mathrm{L}} \frac{|(x'a)| + |(xa)| \cos(\overline{xa}\,\overline{ax'})}{|(xx')|}$$
$$= \underset{x \to x'}{\mathrm{L}} \frac{-|(xx')| \cos(\overline{ax'}\,\overline{x'x})}{|(xx')|},$$

by formula on p. 15,

$$= \underset{x \to x'}{\mathrm{L}} - \cos(\overline{x'a}\,\overline{xx'})$$

$$= -\cos(\tau x \overline{xa}).$$

Hence
$$\frac{d\,|(xa)|}{dx} = -\cos(\tau x \overline{xa}).$$

The line $x_{\tau x_{\frac{\pi}{2}}}$ we shall call the normal line at x, and represent it by νx, so that

$$(\tau x\, \nu x) = \frac{\pi}{2}.$$

Hence
$$\frac{d\,|(xa)|}{dx} = -\cos(\overline{xa}\,\tau x) = -\sin(\overline{xa}\,\nu x)$$

$$= -\frac{(xa\,\nu x)}{|(xa)|} = \frac{(\nu x a)}{|(xa)|}.$$

§ 42. *To differentiate* $(x\alpha)$.

We have
$$\frac{d\,(x\alpha)}{dx} = \underset{x \to x'}{\mathrm{L}} \frac{(x'\alpha) - (x\alpha)}{|(xx')|}$$

$$= \underset{x \to x'}{\mathrm{L}} \frac{(x'x\alpha)}{|(xx')|}$$

$$= -\underset{x \to x'}{\mathrm{L}} \sin(\overline{xx'}\,\alpha)$$

$$= -\sin(\tau x \alpha).$$

§ 43. *To differentiate* (ξa).

The line $p\xi_{\xi_{\frac{\pi}{2}}}$ we shall call the normal line of ξ, and represent it by $\nu\xi$.

We have from definition

$$\frac{d(\xi a)}{d\xi} = \underset{\xi \to \xi'}{\mathrm{L}} \frac{(\xi' a) - (\xi a)}{(\xi\xi')}$$

$$= \underset{\xi \to \xi'}{\mathrm{L}} \frac{-(p\xi a \xi') - (\xi a)}{(\xi\xi')}$$

$$= \underset{\xi \to \xi'}{\mathrm{L}} \frac{-(\xi \nu\xi a \xi') - (\xi a)}{(\xi\xi')}$$

$$= \underset{\xi \to \xi'}{\mathrm{L}} \frac{-(\xi a)\sin(\nu\xi\xi') + (\nu\xi a)\sin(\xi\xi') - (\xi a)}{(\xi\xi')},$$

by formula on p. 18.

$$\therefore \frac{d(\xi a)}{d\xi} = (\nu\xi a).$$

§ 44. *To differentiate* $(\xi\alpha)$.

We have
$$\frac{d(\xi\alpha)}{d\xi}=\underset{\xi\to\xi'}{\mathrm{L}}\frac{(\xi'\alpha)-(\xi\alpha)}{(\xi\xi')}$$
$$=\underset{\xi\to\xi'}{\mathrm{L}}\frac{(\xi'\xi)}{(\xi\xi')}.$$
$$\therefore\ \frac{d(\xi\alpha)}{d\xi}=-1.$$

§ 45. Let $x, y, z \ldots, \xi, \eta, \zeta \ldots$ be a number of points and lines Let $f(x, y, z \ldots, \xi, \eta, \zeta \ldots)$ denote a measure depending on the same points and lines.

Then the differential of $f(x, y, z \ldots, \xi, \eta, \zeta \ldots)$ denoted by $df(x, y, z \ldots, \xi, \eta, \zeta \ldots)$ is defined as the expression
$$f(x', y', z' \ldots, \xi', \eta', \zeta' \ldots)-f(x, y, z \ldots, \xi, \eta, \zeta \ldots),$$
where $x'\ y', z' \ldots, \xi', \eta', \zeta' \ldots$ are near $x, y, z \ldots, \xi, \eta, \zeta \ldots$, small quantities of $\partial x, \partial\xi \ldots$ being only retained.

Then $f(x', y', z' \ldots, \xi', \eta', \zeta' \ldots)-f(x, y, z \ldots, \xi, \eta, \zeta \ldots)$
$$=f(x', y', z' \ldots, \xi', \eta', \zeta' \ldots)-f(x, y', z' \ldots \xi', \eta', \zeta' \ldots)$$
$$+f(x, y', z' \ldots, \xi', \eta', \zeta' \ldots)-f(x, y, z' \ldots, \xi', \eta', \zeta' \ldots)+\ldots$$
$$+f(x, y, z' \ldots, \xi', \eta', \zeta' \ldots)-f(x, y, z \ldots \xi, \eta', \zeta' \ldots)+\ldots.$$

Hence $df(x, y, z \ldots, \xi, \eta, \xi \ldots)$
$$=\frac{\partial f(x, y, z \ldots, \xi, \eta, \zeta \ldots)}{\partial x}dx+\frac{\partial f(x, y, z \ldots, \xi, \eta, \zeta \ldots)}{\partial y}dy$$
$$+\ldots+\frac{\partial f(x, y, z \ldots, \xi, \eta, \zeta \ldots)}{\partial\xi}d\xi+\ldots.$$

§ 46. We have then the following differentiations:
$$d\,|(xy)|=-\cos(\tau x\overline{xy})\,dx-\cos(\tau y\overline{yx})\,dy$$
$$=\frac{(\nu xy)}{|(xy)|}dx+\frac{(\nu yx)}{|(xy)|}dy,$$
$$d(x\eta)=-\sin(\tau x\eta)\,dx+(x\nu\eta)\,d\eta,$$
$$d(\xi\eta)=d\eta-d\xi.$$

§ 47. *Examples.*

1. Shew that $d(xyz)=(yz\tau x)\,dx+(zx\tau y)\,dy+(xy\tau z)\,dz.$

For
$$\frac{\partial(xyz)}{\partial x}=|(yz)|\frac{\partial(x\overline{yz})}{\partial x}=-|(yz)|\sin(\tau x\overline{yz})=(yz\tau x).$$

2. Shew that $d(xy\zeta)=-\sin(\tau x\zeta)\,dx+\sin(\tau y\zeta)\,dy+(xy\nu\zeta)\,d\zeta.$

For
$$\frac{\partial(xy\zeta)}{\partial x}=\frac{\partial(x\zeta)}{\partial x}=-\sin(\tau x\zeta),$$
$$\frac{\partial(xy\zeta)}{\partial\zeta}=\frac{\partial\{(x\zeta)-(y\zeta)\}}{\partial\zeta}=(x\nu\zeta)-(y\nu\zeta)=(xy\,\nu\zeta).$$

3. Shew that $(\xi\eta z)\dfrac{\partial(\xi\eta z)}{\partial\xi}$ = standard measure of the feet of the perpendiculars from z on ξ, $\nu\xi$, η respectively.

We have
$$\tfrac{1}{2}\partial(\xi\eta z)^2=\tfrac{1}{2}\partial\{(\xi z)^2+(\eta z)^2-2(\xi z)(\eta z)\cos(\xi\eta)\}$$
$$=(\xi z)(\nu\xi z)-(\nu\xi z)(\eta z)\cos(\xi\eta)-(\xi z)(\eta z)\sin(\xi\eta)$$
= measure of the feet of the perpendiculars from on ξ, $\nu\xi$, η respectively.

4. Shew that $d(\xi\eta\zeta)=(\nu\xi\eta\zeta)\,d\xi+(\xi\nu\eta\zeta)\,d\eta+(\xi\eta\nu\zeta)\,d\zeta.$

5. Shew that $$\frac{d(\overline{xa}\,b)}{dx}=-|(ab)|\cos(\overline{xa}\,\overline{ab})\frac{(\tau xa)}{(xa)^2}.$$

We have
$$\frac{d(\overline{xa}b)}{dx}=\frac{d}{dx}\frac{(xab)}{|(xa)|}=\frac{(ab\tau x)|(xa)|+(xab)\cos(\tau x\,\overline{xa})}{(xa)^2}$$
$$=[|(xa)(ab)|\sin(\overline{ab}\,\tau x)+(xab)\cos(\tau x\,\overline{xa})]/(xa)^2$$
$$=[|(xa)(ab)|\sin\{(\overline{ab}\,\overline{xa})+(\overline{xa}\,\tau x)\}+(xab)\cos(\tau x\,\overline{xa})]/(xa)^2$$
$$=[|(xa)(ab)|\{\sin(\overline{ab}\,\overline{xa})\cos(\overline{xa}\,\tau x)+\sin(\overline{xa}\,\tau x)\cos(\overline{ab}\,\overline{xa})\}$$
$$+(xab)\cos(\tau x\,\overline{xa})]/(xa)^2$$
$$=|(xa)(ab)|\sin(\overline{xa}\,\tau x)\cos(\overline{xa}\,\overline{ab})/(xa)^2$$
$$=-\frac{(\tau xa)}{(xa)^2}|(ab)|\cos(\overline{xa}\,\overline{ab}).$$

6. Shew that $$\frac{d(\overline{\xi a}\beta)}{d\xi}=-\frac{(p\xi a)\sin(a\beta)}{\sin^2(\xi a)}.$$

$$\frac{d(\overline{\xi a}\beta)}{d\xi}=\frac{d}{d\xi}\frac{(\xi a\beta)}{\sin(\xi a)}=\frac{(\nu\xi a\beta)\sin(\xi a)+(\xi a\beta)\cos(\xi a)}{\sin^2(\xi a)}$$
$$=[(\nu\xi\overline{a\beta})\sin(\xi a)+(\xi a\beta)\cos(\xi a)]\frac{\sin(a\beta)}{\sin^2(\xi a)}$$
$$=-\frac{(p\xi a)\sin(a\beta)}{\sin^2(\xi a)}.$$

7. Shew that $$\frac{d}{dx}(\overline{xa}\beta\gamma)=-\frac{(\tau xa)(a\beta)\sin(\beta\gamma)}{(xa\beta)^2}.$$

We have
$$\frac{d}{dx}(xa\beta\gamma)=(\gamma\beta a\tau x),$$
$$\therefore\ \frac{d}{dx}(\overline{xa}\beta\gamma)=\frac{d}{dx}\frac{(xa\beta\gamma)}{(xa\beta)}$$
$$=\frac{(\gamma\beta a\tau x)(xa\beta)+(xa\beta\gamma)\sin(\tau x\beta)}{(xa\beta)^2}$$
$$=\frac{\{(\gamma a)\sin(\beta\tau x)-(\beta a)\sin(\gamma\tau x)\}\{(x\beta)-(a\beta)\}+\{(x\beta)(a\gamma)-(x\gamma)(a\beta)\}\sin(\tau x\beta)}{(xa\beta)^2}$$
$$=\frac{(\gamma a)(\beta a)\sin(\tau x\beta)-(x\gamma)(a\beta)\sin(\tau x\beta)+(a\beta)\sin(\tau x\gamma)\{(x\beta)-(a\beta)\}}{(xa\beta)^2}$$
$$=\frac{(a\beta)}{(xa\beta)^2}[(\beta\gamma a\tau x)-(\beta\gamma x\tau x)]$$
$$=-\frac{(\tau xa)(a\beta)\sin(\beta\gamma)}{(xa\beta)^2}.$$

CHAPTER V

DIFFERENTIATION OF DETERMINATES OF SIMPLE ELEMENTS

§ 48. Suppose $E(x, y, z \ldots, \xi, \eta, \zeta \ldots)$ denote a determinate of $x, y, z \ldots, \xi, \eta, \zeta \ldots$. Then $E(x', y', z' \ldots, \xi', \eta', \zeta' \ldots)$ denotes a determinate near to $E(x, y, z \ldots, \xi, \eta, \zeta \ldots)$.

We define, as for simple elements,

$$\partial E(x, y, z \ldots, \xi, \eta, \zeta \ldots) = \overline{E(x, y, z \ldots, \xi, \eta, \zeta \ldots) \, E(x', y' \ldots, \xi', \eta' \ldots)}.$$

In general the results are quadratic in the differentials of the several elements.

The method of procedure adopted is as follows:

Suppose, for instance, $E(x, y, z \ldots, \xi, \eta, \zeta \ldots)$ is a point. Call it a.

Then we shall have two measures vanishing, viz.,

$$f_1(x, y, z \ldots, \xi, \eta, \zeta \ldots a) = 0,$$

$$f_2(x, y, z \ldots, \xi, \eta, \zeta \ldots a) = 0.$$

Then we have

$$\frac{\partial f_1}{\partial x} dx + \frac{\partial f_1}{\partial y} dy + \ldots + \frac{\partial f_1}{\partial \xi} d\xi + \frac{\partial f_1}{\partial \eta} d\eta + \ldots + \frac{\partial f_1}{\partial a} da = 0,$$

$$\frac{\partial f_2}{\partial x} dx + \frac{\partial f_2}{\partial y} dy + \ldots + \frac{\partial f_2}{\partial \xi} d\xi + \frac{\partial f_2}{\partial \eta} d\eta + \ldots + \frac{\partial f_2}{\partial a} da = 0.$$

From these two equations and $(a\tau a) = 0$ we eliminate τa, and so find ∂a.

A direct method of procedure will be indicated later.

§ 49. *To differentiate $\overline{xy}$.*

Let $\overline{xy} = \zeta$, then $(x\zeta) = 0$, $(y\zeta) = 0$,

$$\therefore \; -\sin(\tau x\zeta)\, dx + (x\nu\zeta)\, d\zeta = 0,$$

$$-\sin(\tau y\zeta)\, dy + (y\nu\zeta)\, d\zeta = 0.$$

Subtracting,

$$-\sin(\tau x\zeta)\,dx + \sin(\tau y\zeta)\,dy + (xy\nu\zeta)\,d\zeta = 0.$$

Hence $\quad |(xy)|\cos(\overline{xy}\zeta)\,d\zeta = \sin(\tau x\zeta)\,dx - \sin(\tau y\zeta)\,dy,$

$$\therefore\ |(xy)|\,d\overline{xy} = \sin(\tau x\overline{xy})\,dx - \sin(\tau y\overline{xy})\,dy,$$

$$\therefore\ (xy)^2\,d\overline{xy} = (y\tau x)\,dx + (x\tau y)\,dy.$$

NOTE. It will be seen that $d\overline{xy} = -d(\overline{xy}\alpha)$, where α is a fixed line, i.e. the differential of a measure, which explains the linearity of the result.

§ 50. *To differentiate* $\overline{\xi\eta}$.

Let $\overline{\xi\eta} = z$, then $(\xi z) = 0$, $(\eta z) = 0$,

$$\therefore\ -\sin(\tau z\xi)\,dz + (z\nu\xi)\,d\xi = 0,$$

$$-\sin(\tau z\eta)\,dz + (z\nu\eta)\,d\eta = 0.$$

Now since $\qquad (\tau z\xi) + (\xi\eta) + (\eta\tau z) = 2\pi,$

$$\therefore\ \sin^2(\tau z\xi) + \sin^2(\tau z\eta) - 2\sin(\tau z\xi)\sin(\tau z\eta)\cos(\xi\eta) = \sin^2(\xi\eta).$$

Hence $\quad \sin^2(\xi\eta)(dz)^2 = (z\nu\xi)^2(d\xi)^2 + (z\nu\eta)^2(d\eta)^2$

$$- 2(z\nu\xi)(z\nu\eta)\cos(\xi\eta)\,d\xi\,d\eta,$$

$$\therefore\ \sin^4(\xi\eta)(dz)^2 = (\xi\eta\nu\xi)^2(d\xi)^2 + (\xi\eta\nu\eta)^2(d\eta)^2$$

$$- 2(\xi\eta\nu\xi)(\xi\eta\nu\eta)\cos(\xi\eta)\,d\xi\,d\eta,$$

$$\therefore\ \sin^4(\xi\eta)(d\overline{\xi\eta})^2 = (p\xi\eta)^2(d\xi)^2 + (p\eta\xi)^2(d\eta)^2$$

$$+ 2(p\xi\eta)(p\eta\xi)\cos(\xi\eta)\,d\xi\,d\eta.$$

§ 51. *Examples.*

1. Shew that

$$(\xi\eta z)^2\,d\overline{\overline{\xi\eta}z} = (p\xi\eta)(\eta z)\,d\xi + (p\eta\xi)(\xi z)\,d\eta + \sin(\xi\eta)(\xi\eta\tau z)\,dz.$$

Let $\overline{\overline{\xi\eta}z} = \lambda$, then $(\xi\eta\lambda) = 0$, $(z\lambda) = 0$,

$$\therefore\ (\nu\xi\eta\lambda)\,d\xi + (\xi\nu\eta\lambda)\,d\eta + (\xi\eta\nu\lambda)\,d\lambda = 0,$$

$$-\sin(\tau z\lambda)\,dz + (z\nu\lambda)\,d\lambda = 0.$$

Multiplying the second equation by $\sin(\xi\eta)$ and subtracting we have

$$(\nu\xi\eta\lambda)\,d\xi + (\xi\nu\eta\lambda)\,d\eta + \sin(\tau z\lambda)\sin(\xi\eta)\,dz + \sin(\xi\eta)(\overline{\xi\eta}z\nu\lambda)\,d\lambda = 0,$$

$$\therefore\ (\xi\eta z)\,d\overline{\overline{\xi\eta}z} = -(\nu\xi\eta\lambda)\,d\xi - (\xi\nu\eta\lambda)\,d\eta - \sin(\tau z\lambda)\sin(\xi\eta)\,dz$$

$$= -(\nu\xi\eta\overline{\overline{\xi\eta}z})\,d\xi - (\xi\nu\eta\overline{\overline{\xi\eta}z})\,d\eta - \sin(\tau z\overline{\overline{\xi\eta}z})\sin(\xi\eta)\,dz,$$

$$\therefore\ (\xi\eta z)^2\,d\overline{\overline{\xi\eta}z} = (p\xi\eta)(\eta z)\,d\xi + (p\eta\xi)(\xi z)\,d\eta + (\xi\eta\tau z)\sin(\xi\eta)\,dz.$$

2. Shew that

$$(xy\zeta)^4(d\overline{\overline{xy}\zeta})^2 = \{(y\zeta)(y\tau x)\,dx + (x\zeta)(x\tau y)\,dy\}^2 + (xy)^2(xyp\zeta)^2\,d\zeta^2$$

$$+ 2\{(y\zeta)(y\tau x)\,dx + (x\zeta)(x\tau y)\,dy\}(xyp\zeta)(xy\nu\zeta)\,d\zeta.$$

Let $\overline{\overline{xy}\,\zeta}=a$, then $(xya)=0$, $(\zeta a)=0$,

$$\therefore\ (ya\tau x)\,dx+(ax\tau y)\,dy+(xy\tau a)\,da=0,$$
$$(a\nu\zeta)\,d\zeta-\sin(\tau a\zeta)\,da=0.$$

Now since $$(\tau a\zeta)+(\zeta xy)+(\bar{x}\bar{y}\tau a)=2\pi,$$

$$\because\ \sin^2(\tau a\zeta)+\sin^2(\tau a\bar{x}\bar{y})-2\sin(\tau a\zeta)\sin(\tau a\bar{x}\bar{y})\cos(\bar{x}\bar{y}\zeta)=\sin^2(\bar{x}\bar{y}\zeta).$$

$$\therefore\ (xy\zeta)^2=(xy)^2\sin^2(\tau a\zeta)+(xy\tau a)^2+2(xy\tau a)\,|(xy)|\sin(\tau a\zeta)\cos(\bar{x}\bar{y}\zeta),$$

$$\therefore\ (xy\zeta)^2(da)^2=(xy)^2(a\nu\zeta)^2(d\zeta)^2+\{(ya\tau x)\,dx+(ax\tau y)\,dy\}^2$$
$$-2\{(ya\tau x)\,dx+(ax\tau y)\,dy\}\,|(xy)|\cos(\bar{x}\bar{y}\zeta)\,(a\nu\zeta)\,d\zeta,$$

$$\therefore\ (xy\zeta)^4(da)^2=(xy)^2(xy\zeta\nu\zeta)^2(d\zeta)^2+\{-(xy\zeta y\tau x)\,dx+(xy\zeta x\tau y)\,dy\}^2$$
$$-2\{-(xy\zeta y\tau x)\,dx+(xy\zeta x\tau y)\,dy\}\,|(xy)|\cos(\bar{x}\bar{y}\zeta)\,(xy\zeta\nu\zeta)\,d\zeta.$$

Hence $$(xy\zeta)^4(d\overline{\overline{xy}\,\zeta})^2=(xy)^2(xyp\zeta)^2(d\zeta)^2+\{(y\tau x)(\zeta y)\,dx+(x\tau y)(\zeta x)\,dy\}^2$$
$$+2\{(y\tau x)(\zeta y)\,dx+(x\tau y)(\zeta x)\,dy\}\,(xyp\zeta)(xy\nu\zeta)\,d\zeta.$$

3. Shew that

$$(xy\zeta w)^2\,d\overline{\overline{\overline{xy}\,\zeta}\,w}=(w\zeta)(y\zeta)(y\tau x)\,dx+(w\zeta)(x\zeta)(x\tau y)\,dy$$
$$+(xyw)(xyp\zeta)\,d\zeta+(xy\zeta\tau w)\,dw.$$

4. Shew that

$$(\xi\eta z\omega)^4(d\overline{\overline{\overline{\xi\eta}\,z}\,\omega})^2=(\xi\eta z)^2(\xi\eta zp\omega)^2(d\omega)^2$$
$$+\{(\omega z)(\eta z)(p\xi\eta)\,d\xi+(\omega z)(\xi z)(p\eta\zeta)\,d\eta+(\xi\eta\tau z)(\xi\eta\omega)\,dz\}^2$$
$$+2\{(\omega z)(\eta z)(p\xi\eta)\,d\xi+(\omega z)(\xi z)(p\eta\zeta)\,d\eta$$
$$+(\xi\eta\tau z)(\xi\eta\omega)\,dz\}\,(\xi\eta zp\omega)(\xi\eta z\nu\omega)\,d\omega.$$

CHAPTER VI

DIFFERENTIATION OF VECTORIAL ELEMENTS

§ 52. In this Chapter we shall find the differentials of the vectorial elements, $x_{\acute{\rho}\acute{\sigma}\ \ \acute{\omega}}$, $x_{\acute{\rho}\acute{\sigma}\ \ \phi\omega}$. Having found these we may find the differentials of any measures or determinates containing vectorial elements.

§ 53. *To find the value of* $dx_{\acute{\rho}\acute{\sigma}\ldots\acute{\omega}}$.

Take any fixed line λ, then

$$(x_{\acute{\rho}\acute{\sigma}\ .\acute{\omega}}\lambda) = (x\lambda) - \Sigma\hat{\rho}\sin(\rho\lambda),$$

$\therefore$ differentiating

$$\sin(\tau x_{\acute{\rho}\acute{\sigma}\ \ \acute{\omega}}\lambda)\, dx_{\acute{\rho}\acute{\sigma}\ \ \acute{\omega}} = \sin(\tau x\lambda)\, dx$$
$$+ \Sigma\, d\hat{\rho}\sin(\rho\lambda) - \Sigma\,\hat{\rho}\cos(\rho\lambda)\, d\rho;$$

changing λ to $\lambda_{\frac{\pi}{2}}$ we have

$$\sin(\tau x_{\acute{\rho}\acute{\sigma}\ \ \acute{\omega}}\lambda_{\frac{\pi}{2}})\, dx_{\acute{\rho}\acute{\sigma}..\acute{\omega}} = \sin(\tau x\lambda_{\frac{\pi}{2}})\, dx$$
$$+ \Sigma d\hat{\rho}\sin(\rho\lambda_{\frac{\pi}{2}}) - \Sigma\hat{\rho}\cos(\rho\lambda_{\frac{\pi}{2}})\, d\rho,$$

i.e. $$\cos(\tau x_{\acute{\rho}\acute{\sigma}\ .\acute{\omega}}\lambda)\, dx_{\acute{\rho}\acute{\sigma}\ \ \acute{\omega}} = \cos(\tau x\lambda)\, dx$$
$$+ \Sigma\, d\hat{\rho}\cos(\rho\lambda) + \Sigma\,\hat{\rho}\sin(\rho\lambda)\, d\rho.$$

Squaring and adding

$$(dx_{\acute{\rho}\acute{\sigma}\ \ \acute{\omega}})^2 = (dx)^2 + 2dx\Sigma d\hat{\rho}\cos(\tau x\rho) - 2dx\Sigma\hat{\rho}d\rho\sin(\tau x\rho)$$
$$+ \Sigma(d\hat{\rho})^2 + \Sigma\hat{\rho}^2(d\rho)^2 - 2\Sigma(d\hat{\rho}\hat{\sigma}d\sigma - d\hat{\sigma}\hat{\rho}d\rho)\sin(\rho\sigma)$$
$$+ 2\Sigma\,(d\hat{\rho}d\hat{\sigma} + \hat{\rho}\hat{\sigma}d\rho d\sigma)\cos(\rho\sigma).$$

§ 54. *To find the value of* $dx_{\acute{\rho}\acute{\sigma}\ldots\acute{\phi}\omega}$.

Evidently $$dx_{\acute{\rho}\acute{\sigma}\ldots\acute{\phi}\omega} = d\omega.$$

§ 55. *Examples.*

1. Find the value of $d\,|(x_{\acute\rho\acute\sigma}\dots{}_{\acute\omega}y)|$.

$$(x_{\acute\rho\acute\sigma}\dots{}_{\acute\omega}y)^2=(xy)^2-2\,|(xy)|\,\Sigma\hat\rho\cos(\overline{xy}\,\rho)+\Sigma\hat\rho^2+2\Sigma\hat\rho\hat\sigma\cos(\rho\sigma),$$

$$\therefore\ |(x_{\acute\rho\acute\sigma}\dots{}_{\acute\omega}y)|\,d\,|(x_{\acute\rho\acute\sigma}\dots{}_{\acute\omega}y)|=|(xy)|\,d\,|(xy)|-d\,|(xy)|\,\Sigma\,\hat\rho\cos(\overline{xy}\,\rho)$$
$$-|(xy)|\,\Sigma\,\{d\hat\rho\cos(\overline{xy}\,\rho)+\hat\rho\sin(\overline{xy}\,\rho)\,(d\overline{xy}-d\rho)\}$$
$$+\Sigma\,\hat\rho\,d\hat\rho+\Sigma\cos(\rho\sigma)\,(\hat\rho\,d\sigma+\hat\sigma\,d\rho)+\Sigma\,\hat\rho\hat\sigma\sin(\rho\sigma)\,(d\rho-d\sigma)$$

$$=(\nu xy)\,dx+(\nu yx)\,dy-\frac{(\nu xy)\,dx+(\nu yx)\,dy}{|(xy)|}\,\Sigma\,\hat\rho\cos(\overline{xy}\,\rho)$$
$$-|(xy)|\,\Sigma\left[\cos(\overline{xy}\rho)\,d\hat\rho+\hat\rho\sin(\overline{xy}\rho)\left\{\frac{(\tau xy)\,dx+(\tau yx)\,dy}{(xy)^2}-d\rho\right\}\right]$$
$$+\Sigma\hat\rho\,d\hat\rho+\Sigma\,(\hat\rho\,d\hat\sigma+\hat\sigma\,d\hat\rho)\cos(\rho\sigma)+\Sigma\,\hat\rho\hat\sigma\sin(\rho\sigma)\,(d\rho-d\sigma)$$

$$=dx\left\{(\nu xy)+\frac{(\nu xy)}{|(xy)|}\sum_\rho\hat\rho\cos(\overline{xy}\,\rho)+\frac{(\tau xy)}{|(xy)|}\sum_\rho\hat\rho\sin(\overline{xy}\,\rho)\right\}$$
$$+dy\left\{(\nu yx)-\frac{(\nu yx)}{|(xy)|}\sum_\rho\hat\rho\cos(\overline{xy}\rho)-\frac{(\tau yx)}{|(xy)|}\sum_\rho\hat\rho\sin(\overline{xy}\,\rho)\right\}$$
$$+\sum_\rho d\hat\rho\,\{-|(xy)|\cos(\overline{xy}\,\rho)+\hat\rho+\sum_\sigma\hat\sigma\cos(\rho\sigma)\}$$
$$+\sum_\rho\hat\rho\,d\rho\,\{|(xy)|\sin(\overline{xy}\,\rho)+\sum_\sigma\hat\sigma\sin(\rho\sigma)\}$$

$$=dx\,\{(\nu xy)+\sum_\rho\hat\rho\cos(\tau x\rho)\}+dy\,\{(\nu yx)+\sum_\rho\hat\rho\cos(\tau y\rho)\}$$
$$+\sum_\rho d\hat\rho\,\{-(xy\nu\rho)+\hat\rho+\sum_\sigma\hat\sigma\cos(\rho\sigma)\}$$
$$+\sum_\rho\hat\rho\,d\rho\,\{(xy\rho)+\sum_\sigma\hat\sigma\sin(\rho\sigma)\}.$$

2. Find the value of $d\overline{x_{\acute\rho\acute\sigma}\dots{}_{\acute\omega}y}$.

Let $\overline{x_{\acute\rho\acute\sigma}\dots{}_{\acute\omega}y}=\zeta$ so that $(x_{\acute\rho\acute\sigma}\dots{}_{\acute\omega}\zeta)=0,\ (y\zeta)=0.$

$$\therefore\ (x\zeta)-\hat\rho\sin(\rho\zeta)-\hat\sigma\sin(\sigma\zeta)-\dots=0,$$

$$\therefore\ -\sin(\tau x\zeta)\,dx+(x\nu\zeta)\,d\zeta-\Sigma d\hat\rho\sin(\rho\zeta)+\Sigma\,\hat\rho\cos(\rho\zeta)\,(d\rho-d\zeta)=0,$$

and
$$-\sin(\tau y\zeta)\,dy+(y\nu\zeta)\,d\zeta=0,$$

$\therefore$ subtracting

$$-\sin(\tau x\zeta)\,dx+\sin(\tau y\zeta)\,dy+(xy\zeta_{\frac{\pi}{2}})\,d\zeta-\Sigma\,d\hat\rho\sin(\rho\zeta)$$
$$+\Sigma\,\hat\rho\,d\rho\cos(\rho\zeta)-d\zeta\,\Sigma\,\hat\rho\cos(\rho\sigma)=0,$$

$$\therefore\ d\zeta\,[(xy\zeta_{\frac{\pi}{2}})-\Sigma\,\hat\rho\cos(\rho\sigma)]=\sin(\tau x\zeta)\,dx-\sin(\tau y\zeta)\,dy$$
$$+\Sigma\,d\hat\rho\sin(\rho\zeta)-\Sigma\,\hat\rho\,d\rho\cos(\rho\zeta),$$

$$d\zeta\,[|(xy)|\cos(\overline{x_{\acute\rho\acute\sigma}\dots{}_{\acute\omega}y}\,\overline{xy})-\Sigma\hat\rho\cos(\overline{x_{\acute\rho\acute\sigma}\dots{}_{\acute\omega}y}\,\rho)]$$
$$=-\sin(\overline{x_{\acute\rho\acute\sigma}\dots{}_{\acute\omega}y}\,\overline{\tau x})\,dx+\sin(\overline{x_{\acute\rho\acute\sigma}\dots{}_{\acute\omega}y}\,\overline{\tau y})\,dy$$
$$-\Sigma\,d\hat\rho\sin(\overline{x_{\acute\rho\acute\sigma}\dots{}_{\acute\omega}y}\,\rho)-\Sigma\,\hat\rho\,d\rho\cos(\overline{x_{\acute\rho\acute\sigma}\dots{}_{\acute\omega}y}\,\rho),$$

$$\therefore\ d\zeta\,[|(xy)|\,(x_{\acute\rho\acute\sigma}\dots{}_{\acute\omega}y\,xy_{\frac{\pi}{2}})-\Sigma\hat\rho(x_{\acute\rho\acute\sigma}\dots{}_{\acute\omega}y\nu\rho)]$$
$$=-(x_{\acute\rho\acute\sigma}\dots{}_{\acute\omega}y\,\tau x)\,dx+(x_{\acute\rho\acute\sigma}\dots{}_{\acute\omega}y\,\tau y)\,dy$$
$$-\Sigma\,d\hat\rho\,(x_{\acute\rho\acute\sigma}\dots{}_{\acute\omega}y\rho)-\Sigma\,\hat\rho\,d\rho\,(x_{\acute\rho\acute\sigma}\dots{}_{\acute\omega}y\,\nu\rho),$$

$$\therefore\ d\zeta\left[\,|(xy)|\,\{(xy\,\overline{xy}_{\frac{\pi}{2}})-\Sigma\,\hat{\rho}\sin(\rho\,\overline{xy}_{\frac{\pi}{2}})\}-\Sigma\,\hat{\rho}\,\{(xy\,\nu\rho)-\Sigma\,\hat{\sigma}\sin(\sigma\nu\rho)\}\right]$$

$$=\{(y\tau x)+\Sigma\,\hat{\rho}\sin(\rho\tau x)\}\,dx+\{(x\tau y)-\Sigma\,\hat{\rho}\sin(\rho\tau y)\}\,dy$$

$$-\Sigma\,d\hat{\rho}\,\{(xy\rho)+\sum_{\sigma}\hat{\sigma}\sin(\rho\sigma)\}-\Sigma\,\hat{\rho}\,d\rho\,\{(xy\nu\rho)-\hat{\rho}-\sum_{\sigma}\hat{\sigma}\cos(\rho\sigma)\},$$

$$\therefore\ (x_{\hat{\rho}\hat{\sigma}}\ \ _{\hat{\omega}}y)^2\,d\,\overline{x_{\hat{\rho}\hat{\sigma}}\ \ .\hat{\omega}y}$$

$$=\{(y\tau x)+\Sigma\,\hat{\rho}\sin(\rho\tau x)\}\,dx+\{(x\tau y)-\Sigma\,\hat{\rho}\sin(\rho\tau y)\}\,dy$$

$$-\Sigma\,(xy\rho)\,d\hat{\rho}-\Sigma\,(xy\nu\rho)\,\hat{\rho}\,d\rho+\Sigma\,\hat{\rho}^2\,d\rho+\Sigma\,\hat{\rho}\hat{\sigma}\,(d\rho+d\sigma)\cos(\rho\sigma)$$

$$+\Sigma\,(\hat{\rho}\,d\hat{\sigma}-\hat{\sigma}\,d\hat{\rho})\sin(\rho\sigma).$$

Corollary. $\{\Sigma\,\hat{\rho}^2+2\Sigma\,\hat{\rho}\hat{\sigma}\cos(\rho\sigma)\}\,d\,\overline{xx}_{\hat{\rho}\hat{\sigma}}.\ _{\hat{\omega}}$

$$=\Sigma\,\hat{\rho}^2\,d\rho+\Sigma\,\hat{\rho}\hat{\sigma}\,(d\rho+d\sigma)\cos(\rho\sigma)+\Sigma\,(\hat{\rho}\,d\hat{\sigma}-\hat{\sigma}\,d\hat{\rho})\sin(\rho\sigma).$$

3. Find the value of $\qquad d\,\bar{x}_{\hat{\rho}\hat{\sigma}}\ \ \bar{\hat{\phi}}\bar{\omega}\,\eta.$

Let $\qquad \bar{x}_{\hat{\rho}\hat{\sigma}}\ \ \bar{\hat{\phi}}\bar{\omega}\,\eta=z.$

Then $\qquad (x_{\hat{\rho}\hat{\sigma}}\ \ \phi_\omega z)=0,\ \ (\eta z)=0,$

$$\therefore\ (zx\omega)+\Sigma\,\hat{\rho}\sin(\rho\omega)=0,$$

$$\therefore\ -\sin(\tau z\omega)\,dz+\sin(\tau x\omega)\,dx+(zx\nu\omega)\,d\omega$$

$$+\Sigma\,d\hat{\rho}\sin(\rho\omega)-\Sigma\,\hat{\rho}\cos(\rho\omega)\,(d\rho-d\omega)=0,$$

$$\therefore\ \sin(\tau z\omega)\,dz=\sin(\tau x\omega)\,dx+\{(zx\nu\omega)+\Sigma\,\hat{\rho}\cos(\rho\omega)\}\,d\omega$$

$$+\Sigma\,d\hat{\rho}\sin(\rho\omega)-\Sigma\,\hat{\rho}\,d\rho\cos(\rho\omega)$$

and $\qquad \sin(\tau z\eta)\,dz=(\nu\eta z)\,d\eta.$

Now we have

$$\sin^2(\tau z\omega)+\sin^2(\tau z\eta)-2\sin(\tau z\omega)\sin(\tau z\eta)\cos(\omega\eta)=\sin^2(\omega\eta),$$

$$\therefore\ (dz)^2\sin^2(\omega\eta)=[\sin(\tau x\omega)\,dx+\{(zx\nu\omega)+\Sigma\,\hat{\rho}\cos(\rho\omega)\}\,d\omega$$

$$+\Sigma\,d\hat{\rho}\sin(\rho\omega)-\Sigma\,\hat{\rho}\,d\rho\cos(\rho\omega)]^2+(\nu\eta z)^2\,d\eta^2$$

$$-2\,(\nu\eta z)\,d\eta\,[\sin(\tau x\omega)\,dx+\{(zx\nu\omega)+\Sigma\,\hat{\rho}\cos(\rho\omega)\}\,d\omega$$

$$+\Sigma\,d\hat{\rho}\sin(\rho\omega)-\Sigma\,\hat{\rho}\,d\rho\cos(\rho\omega)]\cos(\eta\omega)$$

$$=[\sin(\tau x\omega)\,dx+\{(\overline{x_{\hat{\rho}\hat{\sigma}}\ \ \phi_\omega\eta}\,x\nu\omega)+\Sigma\,\hat{\rho}\cos(\rho\omega)\}\,d\omega+\Sigma\,d\hat{\rho}\sin(\rho\omega)$$

$$-\Sigma\,\hat{\rho}\,d\rho\cos(\rho\omega)]^2+(\nu\eta z)^2\,d\eta^2$$

$$-2\,(\nu\eta z)\,d\eta\cos(\eta\omega)\,[\sin(\tau x\omega)\,dx+\{(x_{\hat{\rho}\hat{\sigma}}\ \ \phi_\omega\eta.x\nu\omega)+\Sigma\,\hat{\rho}\cos(\rho\omega)\}\,d\omega$$

$$+\Sigma\,d\hat{\rho}\sin(\rho\omega)-\Sigma\,\hat{\rho}\,d\rho\cos(\rho\omega)]$$

$$\sin^4(\omega\eta)\,(d\,\bar{x}_{\hat{\rho}\hat{\sigma}}\ .\ \bar{\phi}_\omega\eta)^2=[\sin(\tau x\omega)\sin(\omega\eta)\,dx+\{-(x\eta)+\Sigma\,\hat{\rho}\sin(\rho\eta)\}\,d\omega$$

$$+\Sigma\,d\hat{\rho}\sin(\rho\omega)-\Sigma\,\hat{\rho}\,d\rho\cos(\rho\omega)]^2+(\nu\eta z)^2\,d\eta^2$$

$$+2\,(\nu\eta z)\,d\eta\cos(\eta\omega)\sin(\eta\omega)\,[\sin(\tau x\omega)\sin(\omega\eta)\,dx$$

$$+\{-(x\eta)+\Sigma\,\hat{\rho}\sin(\rho\eta)\}\,d\omega+\Sigma\,d\hat{\rho}\sin(\rho\omega)-\Sigma\,\hat{\rho}\,d\rho\cos(\rho\omega)].$$

4. Reduce $\qquad \overline{\overline{x_\omega\eta z}}$

Let $\qquad \overline{\overline{x_\omega\eta z}}=\lambda,$

then $\qquad (x_\omega\eta\lambda)=0,\ \ (\eta z)=0.$

$$d\,(x_\omega\eta\lambda)=\sin(\eta\lambda)\,d\,(x_\omega\overline{\eta\lambda})_{\eta,\,\lambda\text{ const.}}+(x_\omega\nu\eta\lambda)\,d\eta+(x_\omega\eta\nu\lambda)\,d\lambda$$

$$=\sin(\eta\lambda)\,d\,(\overline{\eta\lambda}\,x\omega)_{\eta,\,\lambda\text{ const.}}+(x_\omega\nu\eta\lambda)\,d\eta+(x_\omega\eta\nu\lambda)\,d\lambda,$$

$$\therefore\ \sin(\eta\lambda)\sin(\tau x\omega)\,dx + (\eta\lambda . x\omega_{\frac{\pi}{2}})\,d\omega + (x_\omega\nu\eta\lambda)\,d\eta + (x_\omega\eta\nu\lambda)\,d\lambda = 0,$$

and $$\sin(\tau z\lambda)\,dz - (z\nu\lambda)\,d\lambda = 0,$$

multiplying second by $\sin(\omega\eta)$ and adding

$$\sin(\eta\lambda)\sin(\tau x\omega)\,dx + (\eta\lambda x\omega_{\frac{\pi}{2}})\,d\omega + (x_\omega\nu\eta\lambda)\,d\eta$$
$$+\sin(\tau z\lambda)\sin(\omega\eta)\,dz + (x_\omega\,\eta z\nu\lambda)\,d\lambda = 0.$$

$$\therefore\ -(x_\omega\eta z\eta)\sin(\tau x\omega)\,dx - (x_\omega\,\eta z\,\eta x\omega_{\frac{\pi}{2}})\,d\omega$$
$$+(x_\omega\,\eta z x_\omega\,\nu\eta)\,d\eta - (x_\omega\eta z\tau z)\sin(\omega\eta)\,dz + (x_\omega\eta z)^2\,d\,\overline{\overline{x_\omega\eta}z} = 0.$$

$$\therefore\ (x_\omega\eta z)^2\,d\,\overline{\overline{x_\omega\eta}z} = -\sin(\tau x\omega)(\eta z)\sin(\omega\eta)\,dx$$
$$+(\eta z)(\eta x)\,d\omega + (p\eta x\omega)(zx\omega)\,d\eta + (x_\omega\eta\tau z)\,dz.$$

CHAPTER VII

DIFFERENTIATION OF EQUATIONAL ELEMENTS

§ 56. As we discussed the differentials of vectorial elements, so we shall in a similar manner discuss the differentials of equational elements.

§ 57. *To find the value of* $d\,\{\Sigma a_r(x\alpha_r)+a=0\}$.

We have from formula of § 30

$$\sin(\xi\beta)\,\Sigma a_r\cos(\alpha_r\beta)=\cos(\xi\beta)\,\Sigma a_r\sin(\alpha_r\beta),$$

where ξ stands for $\Sigma a_r(x\alpha_r)+a=0$.

Differentiating, with β fixed,

$$\begin{aligned}&-d\xi\cos(\xi\beta)\,\Sigma a_r\cos(\alpha_r\beta)\\ &\qquad\qquad+\sin(\xi\beta)\,\{\Sigma da_r\cos(\alpha_r\beta)+\Sigma a_r\sin(\alpha_r\beta)\,d\alpha_r\}\\ &=d\xi\sin(\xi\beta)\,\Sigma a_r\sin(\alpha_r\beta)\\ &\qquad\qquad+\cos(\xi\beta)\,\{\Sigma da_r\sin(\alpha_r\beta)-\Sigma a_r\cos(\alpha_r\beta)\,d\alpha_r\},\end{aligned}$$

$$\begin{aligned}\therefore\ d\xi\,&\{\sin(\xi\beta)\,\Sigma a_r\sin(\alpha_r\beta)+\cos(\xi\beta)\,\Sigma a_r\cos(\alpha_r\beta)\}\\ &=\Sigma da_r\,\{\sin(\xi\beta)\cos(\alpha_r\beta)-\cos(\xi\beta)\sin(\alpha_r\beta)\}\\ &\qquad+\Sigma a_r d\alpha_r\,\{\sin(\xi\beta)\sin(\alpha_r\beta)+\cos(\xi\beta)\cos(\alpha_r\beta)\}\end{aligned}$$

$$d\xi\,\Sigma a_r\cos(\xi\alpha_r)=\Sigma da_r\sin(\xi\alpha_r)+\Sigma a_r d\alpha_r\cos(\xi\alpha_r),$$

$$\therefore\ d\xi=\frac{\sum_r da_r\sum_s a_s\sin(\alpha_s\alpha_r)+\sum_r a_r d\alpha_r\sum_s a_s\cos(\alpha_s\alpha_r)}{\sum_r a_r\sum_s a_s\cos(\alpha_r\alpha_s)}$$

$$=\frac{\Sigma(a_s da_r-a_r da_s)\sin(\alpha_s\alpha_r)+\Sigma a_r^2\,d\alpha_r+\sum_{r\neq s}a_r a_s\cos(\alpha_r\alpha_s)(d\alpha_r+d\alpha_s)}{\Sigma a_r^2+2\,\Sigma a_r a_s\cos(\alpha_r\alpha_s)}.$$

$$\therefore\ d\,\{\Sigma a_r(x\alpha_r)+a=0\}$$

$$=\frac{\Sigma(a_r da_s-a_s da_r)\sin(\alpha_r\alpha_s)+\Sigma a_r^2 d\alpha_r+\sum_{r\neq s}a_r a_s\cos(\alpha_r\alpha_s)(d\alpha_r+d\alpha_s)}{\Sigma a_r^2+2\,\Sigma a_r a_s\cos(\alpha_r\alpha_s)}$$

§ 58. *To find the value of* $d\,\{\Sigma A_r(\xi a_r) + \Sigma B_r \cos(\xi\beta_r) = 0\}$.

We have from formula of § 34

$$(\Sigma A_r(\xi a_r) + \Sigma B_r \cos(\xi\beta_r) = 0 \;\; \gamma) = \frac{\Sigma A_r(\gamma a_r) + \Sigma B_r \cos(\gamma\beta_r)}{\Sigma A_r}.$$

Differentiating, with γ constant, we have writing a for the equational point

$$\begin{aligned}\sum_r A_r . \sin(\tau a\gamma)\, da &= \sum_r |(aa_r)| \sin(\overline{aa_r}\gamma)\, dA_r + \sum_r A_r \sin(\tau a_r\gamma)\, da_r \\ &\quad - \sum_r dB_r \cos(\gamma\beta_r) + \sum_r B_r \sin(\gamma\beta_r)\, d\beta_r.\end{aligned}$$

Changing γ to $\gamma_{\frac{\pi}{2}}$,

$$\begin{aligned}\sum_r A_r . \cos(\tau a\gamma)\, da &= \sum_r |(aa_r)| \cos(\overline{aa_r}\gamma)\, dA_r + \sum_r A_r \cos(\tau a_r\gamma)\, da_r \\ &\quad - \sum_r dB_r \sin(\gamma\beta_r) - \sum_r B_r \cos(\gamma\beta_r)\, d\beta_r.\end{aligned}$$

Squaring and adding,

$$\begin{aligned}(da)^2 . (\sum_r A_r)^2 &= \sum_r (aa_r)^2 (dA_r)^2 + \sum_r A_r^2 (da_r)^2 + \sum_r (dB_r)^2 + \sum_r B_r^2 (d\beta_r)^2 \\ &\quad + 2 \sum_{r \neq s} |(aa_r)(aa_s)|\, dA_r dA_s \cos(\overline{aa_r}, \overline{aa_s}) \\ &\quad + 2 \sum_{r \neq s} A_r A_s \cos(\tau a_r\, \tau a_s)\, da_r da_s \\ &\quad + 2 \sum_{r \neq s} dB_r dB_s \cos(\beta_r \beta_s) \\ &\quad + 2 \sum_{r \neq s} B_r B_s d\beta_r d\beta_s \cos(\beta_r \beta_s) \\ &\quad + 2 \sum_{r, s} dA_r A_s da_s (aa_r \nu a_s) \\ &\quad - 2 \sum_{r, s} dA_r dB_s (aa_r \beta_s) \\ &\quad - 2 \sum_{r, s} dA_r B_s d\beta_s (aa_r \nu \beta_s) \\ &\quad - 2 \sum_{r, s} A_r da_r dB_s \sin(\tau a_r \beta_s) \\ &\quad - 2 \sum_{r, s} A_r da_r B_s d\beta_s \cos(\tau a_r \beta_s) \\ &\quad - 2 \sum_{r, s} dB_r B_s d\beta_s \sin(\beta_r \beta_s).\end{aligned}$$

Hence $(da)^2 . (\sum_r A_r)^4$

$$= \sum_r (dA_r)^2 \{\sum_h A_h^2 (a_r a_h)^2 + 2 \sum_{h \neq k} A_h A_k |(a_r a_h)(a_r a_k)| \cos (\overline{a_r a_h}\,\overline{a_r a_k})$$
$$+ 2 \sum_{h,k} A_h B_k (a_h a_r \beta_k) + \sum_h B_h^2 + 2 \sum_{h \neq k} B_h B_k \cos (\beta_h \beta_k)\}$$
$$+ \sum_{r \neq s} dA_r dA_s \{-(a_r a_s)^2 (\sum_h A_h)^2 + \sum_h A_h^2 (a_r a_h)^2$$
$$+ \sum_{h \neq k} A_h A_k |(a_r a_h)(a_r a_k)| \cos (\overline{a_r a_h}\,\overline{a_r a_k})$$
$$+ 2 \sum_{h,k} A_h B_k (a_h a_r \beta_k) + 2 \sum_h B_h^2 + 4 \sum_{h \neq k} B_h B_k \cos (\beta_h \beta_k)$$
$$+ \sum_h A_h^2 (a_r a_h)^2 + \sum_{h \neq k} A_h A_k |(a_s a_h)(a_s a_k)| \cos (\overline{a_s a_h}\,\overline{a_s a_k})$$
$$+ 2 \sum_{h,k} A_h B_k (a_h a_s \beta_k)\}$$
$$+ 2 \sum_h A_h . \sum_{r,s} dA_r A_s da_s \{\sum_h A_h (a_h a_r \nu a_s) - \sum_h B_h \sin (\beta_h \nu a_s)\}$$
$$- 2 \sum_h A_h . \sum_{r,s} dA_r dB_s \{\sum_h A_h (a_h a_r \beta_s) + \sum_h B_h \cos (\beta_h \beta_s)\}$$
$$- 2 \sum_h A_h . \sum_{r,s} dA_r B_s d\beta_s \{\sum_h A_h (a_h a_s \nu \beta_s) - \sum_h B_h \sin (\beta_h \beta_s)\}$$
$$+ (\sum_h A_h)^2 [\sum_r (dB_r)^2 + \sum_r B_r^2 (d\beta_r)^2 + 2 \sum_{r \neq s} A_r A_s \cos (\tau a_r \tau a_s)\, da_r da_s$$
$$+ 2 \sum_{r \neq s} dB_r dB_s \cos (\beta_r \beta_s) + 2 \sum_{r \neq s} B_r B_s d\beta_r d\beta_s \cos (\beta_r \beta_s)$$
$$- 2 \sum_{r,s} A_r da_r dB_s \sin (\tau a_r \beta_s) - \sum_{r,s} A_r da_r B_s d\beta_r \cos (\tau a_r \beta_s)$$
$$- 2 \sum_{r,s} dB_r B_s d\beta_s \sin (\beta_r \beta_s)].$$

§ 59. *Examples.*

1. Reduce $\qquad d \Sigma \overline{a_r (xa_r) + a = 0}\ \beta.$

Let $\qquad \Sigma a_r (xa_r) + a = 0\ \beta = z.$

Then $\qquad (\Sigma a_r (xa_r) + a = 0\ z) = 0, \ (\beta z) = 0.$

$$\therefore\ \Sigma a_r (za_r) + a = 0, \ (\beta z) = 0,$$
$$\therefore\ -\sin (\tau z \beta)\, dz + (z \nu \beta)\, d\beta = 0.$$
$$\therefore\ \Sigma a_r \sin \{(\tau z \beta) - (a_r \beta)\}\, dz = \Sigma da_r (za_r) + \Sigma a_r (z \nu a_r)\, da_r + da.$$
$$\therefore\ \sin (\tau z \beta)\, dz \Sigma a_r \cos (a_r \beta) - \cos (\tau z \beta)\, dz\, \Sigma a_r \sin (a_r \beta)$$
$$= \Sigma da_r (za_r) + \Sigma a_r (z \nu a_r)\, da_r + da.$$
$$\therefore\ \cos (\tau z \beta)\, dz\, \Sigma a_r \sin (a_r \beta)$$
$$= (z \nu \beta)\, d\beta\, \Sigma a_r \cos (a_r \beta) - \Sigma da_r (za_r) - \Sigma a_r (z \nu a_r)\, da_r - da,$$
$$\therefore\ (dz)^2 \{\Sigma a_r \sin (a_r \beta)\}^2 = \{(z \nu \beta)\, d\beta\}^2 \{\Sigma a_r \sin (a_r \beta)\}^2$$
$$+ \{(z \nu \beta)\, d\beta\, \Sigma a_r \cos (a_r \beta) - \Sigma da_r (za_r) - \Sigma a_r (z \nu a_r)\, da_r - da\}^2$$
$$= (z \nu \beta)^2 d\beta^2 \{\Sigma a_r^2 + 2 \Sigma a_r a_s \cos (a_r a_s)\}$$
$$+ \{\Sigma da_r (za_r) + \Sigma a_r (z \nu a_r)\, da_r + da\}^2$$
$$- 2 (z \nu \beta)\, d\beta\, \Sigma a_r \cos (a_r \beta) \{\Sigma da_r (za_r) + \Sigma a_r (z \nu a_r)\, da_r + da\},$$

substituting for $z = \Sigma \overline{a_r (xa_r) + a = 0}\ \beta$ we get the value of $(dz)^2$.

2. Reduce $$d\overline{(xa)-k=0\ \beta}.$$

We have, putting $$\overline{(xa)-k=0\ \beta}=z,$$

$$(za)=k,\ (z\beta)=0,$$

$$\therefore\ -\sin(\tau za)\,dz+(z\nu a)\,da=dk,$$

$$-\sin(\tau z\beta)\,dz+(z\nu\beta)\,d\beta=0.$$

Now $\sin^2(\tau za)+\sin^2(\tau z\beta)-2\sin(\tau za)\sin(\tau z\beta)\cos(a\beta)=\sin^2(a\beta)$,

$$\therefore\ \sin^2(a\beta)(dz)^2=\{(z\nu a)\,da-dk\}^2+(z\nu\beta)^2d\beta^2-2(z\nu\beta)\,d\beta\,\{(z\nu a)\,da-dk\}\cos(a\beta)$$

$$=(z\nu a)^2da^2+(z\nu\beta)^2d\beta^2-2(z\nu a)(z\nu\beta)\,da\,d\beta\cos(a\beta)-2(z\nu a)\,da\,dk+2(z\nu\beta)\,d\beta\,dk\cos(a\beta)+dk^2.$$

$$\therefore\ \sin^4(a\beta)(dz)^2=((xa)-k=0\ \beta\nu a)^2da^2+((xa)-k=0\ \beta\nu\beta)^2d\beta^2$$
$$-2((xa)-k=0\ \beta\nu a)((xa)-k=0\ \beta\nu\beta)\,da\,d\beta\cos(a\beta)$$
$$-2da\,dk((xa)-k=0\ \beta\nu a)+2d\beta\,dk((xa)-k=0\ \beta\nu\beta)\cos(a\beta)+dk^2,$$

$$\therefore\ \sin^4(a\beta)(dz)^2=\{(pa\beta)+k\cos(a\beta)\}^2da^2+\{(p\beta a)-k\}^2d\beta^2$$
$$-2\{(pa\beta)+k\cos(a\beta)\}\{(p\beta a)-k\}\,da\,d\beta\cos(a\beta)$$
$$+2\,da\,dk\,\{(pa\beta)+k\cos(a\beta)\}+2d\beta\,dk\,\{(p\beta a)-k\}\cos(a\beta)+dk^2.$$

Reduce $$d\overline{\Sigma A_r(\xi a_r)+\Sigma B_r\cos(\xi\beta_r)=0\ c}.$$

Let $$\overline{\Sigma A_r(\xi a_r)+\Sigma B_r\cos(\xi\beta_r)=0\ c}=\lambda,$$

$$\therefore\ \Sigma A_r(\lambda a_r)+\Sigma B_r\cos(\lambda\beta_r)=0,\ (c\lambda)=0,$$

$$\therefore\ \Sigma dA_r(\lambda a_r)+\Sigma A_r(\nu\lambda a_r)\,d\lambda-\Sigma A_r\sin(\tau a_r\lambda)\,da_r$$
$$+\Sigma dB_r\cos(\lambda\beta_r)+\Sigma B_r\sin(\lambda\beta_r)(d\lambda-d\beta_r)=0$$
$$-\sin(\tau c\lambda)\,dc+(c\nu\lambda)\,d\lambda=0.$$

Multiplying the second by ΣA_r and subtracting

$$\Sigma dA_r(\lambda a_r)+\Sigma A_r d\lambda(a_r c\nu\lambda)-\Sigma A_r\sin(\tau a_r\lambda)\,da_r$$
$$+\Sigma dB_r\cos(\lambda\beta_r)+d\lambda\Sigma B_r\sin(\lambda\beta_r)-\Sigma B_r\sin(\lambda\beta_r)\,d\beta_r$$
$$+\Sigma A_r\sin(\tau c\lambda)\,dc=0$$

Whence

$$\Sigma A_r(\xi a_r)+\Sigma B_r\cos(\xi\beta_r)=0\ c)^2\,d\,\overline{\Sigma A_r(\xi a_r)+\Sigma B_r\cos(\xi\beta_r)=0\ c}$$
$$=-\sum_{r,s}(a_r a_s c)(A_r dA_s-A_s dA_r)+\sum_{r,s}dA_r B_s(ca_r\nu\beta_s)$$
$$-\sum_r A_r^2 da_r(\tau a_r c)+\sum_{r\neq s}A_r A_s\{da_r(a_s c\tau a_r)+da_s(a_r c\tau a_s)\}$$
$$+\sum_{r,s}A_r B_s da_r\cos(\tau a_r\beta_s)+\sum_{r,s}dB_r A_s(a_s c\nu\beta_r)$$
$$-\sum_{r,s}(B_r dB_s-B_s dB_r)\sin(\beta_r\beta_s)-\sum_{r\neq s}B_r B_s(d\beta_r+d\beta_s)\cos(\beta_r\beta_s)$$
$$-\sum_r B_r^2 d\beta_r-\sum_{r,s}A_s B_r d\beta_r(a_s c\beta_r)$$
$$-dc\,\Sigma A_r\{\Sigma A_r(\tau ca_r)+\Sigma B_r\cos(\tau c\beta_r)\}.$$

CHAPTER VIII

REDUCTION OF MEASURES CONTAINING FUNCTIONAL ELEMENTS

§ 60. We give in this chapter formulae which enable us to evaluate any measure containing any of the functional elements defined by p; ν, τ.

We require the values of

$$(\tau x a), (\tau x \alpha); \; (\nu x a), (\nu x \alpha); \; (p \xi a), (p \xi \alpha); \; (\nu \xi a), (\nu \xi \alpha).$$

§ 61. *Formula for the evaluation of* $(\tau x a)$.

We have from § 49

$$(\tau x a) = (x a)^2 \frac{d \overline{x a}}{d x}.$$

Formula for the evaluation of $(\tau x \alpha)$.

We have $$\sin (\tau x \alpha) = - \frac{d (x \alpha)}{d x}.$$

Formula for the evaluation of $(\nu x a)$.

We have $$(\nu x a) = \frac{1}{2} \frac{d (x a)^2}{d x} \text{ from § 46.}$$

Formula for the evaluation of $(\nu x \alpha)$.

We have $$\cos (\nu x \alpha) = - \frac{d (x \alpha)}{d x}.$$

§ 62. *Formula for the evaluation of* $(p \xi a)$.

We have $$(\nu \xi a) = \frac{d (\xi a)}{d \xi}.$$

But $$(\xi a)^2 + (\nu \xi a)^2 = (p \xi a)^2,$$

$$\therefore \; (p \xi a)^2 = (\xi a)^2 + \left\{ \frac{d (\xi a)}{d \xi} \right\}^2.$$

Formula for the evaluation of $(p\xi\alpha)$.

We have $(p\xi\alpha)=(\xi\nu\xi\alpha)=(\nu\xi\alpha\xi)=\sin(\alpha\xi)(\nu\xi\overline{\alpha\xi})$.

Now $$(\nu\xi\overline{\alpha\xi})=\left[\frac{\partial(\xi a)}{\partial\xi}\right]_{a=\overline{\alpha\xi}}.$$

Hence $$(p\xi\alpha)=\sin(\alpha\xi)\left[\frac{\partial(\xi a)}{\partial\xi}\right]_{a=\overline{\alpha\xi}},$$

$|(p\xi\alpha)|$ may also be found from the formula

$$|(p\xi\alpha)|=\sin^2(\xi\alpha)\frac{d\overline{\xi\alpha}}{|d\xi|},$$

but the sign of $(p\xi\alpha)$ cannot be determined from this.

We may also evaluate $(p\xi\alpha)$ from

$$(p\xi\alpha)=-\frac{\sin^2(\xi\alpha)}{\sin(\alpha\beta)}\frac{d}{d\xi}(\overline{\xi\alpha}\beta).$$ See Ex. 6, p. 42.

Formula for the evaluation of $(\nu\xi a)$.

We have seen $$(\nu\xi a)=\frac{d(\xi a)}{d\xi}.$$

Formula for the evaluation of $(\nu\xi\alpha)$.

We have $$(\nu\xi\alpha)=(\xi\alpha)-\frac{\pi}{2}.$$

§ 63. *Examples.*

1. Reduce $(\tau\overline{\xi\eta}\,a)$.

We have from § 51, Ex. 1,

$$(\xi\eta a)^2\,d\,\overline{\overline{\xi\eta}\,a}=(p\xi\eta)(\eta a)\,d\xi+(p\eta\xi)(\xi a)\,d\eta,$$

$$\therefore\ (\tau\xi\eta a)=\frac{(\xi\eta a)^2\,d\,\overline{\overline{\xi\eta}\,a}}{\sin^2(\xi\eta)\,d\,\overline{\xi\eta}}$$
$$=\frac{(p\xi\eta)(\eta a)\,d\xi+(p\eta\xi)(\xi a)\,d\eta}{|\sqrt{(p\xi\eta)^2\,d\xi^2+(p\eta\xi)^2\,d\eta^2+2\,(p\xi\eta)(p\eta\xi)\cos(\xi\eta)\,d\xi\,d\eta}|}.$$

2. Reduce $\sin(\tau\overline{\xi\eta}\,a)$.

$$\sin(\tau\overline{\xi\eta}\,a)=-\left[\frac{d(\overline{\xi\eta}\,a)}{d\overline{\xi\eta}}\right]_{a\ \mathrm{const.}}$$
$$=\frac{(p\xi\eta)\sin(\eta a)\,d\xi+(p\eta\xi)\sin(\xi a)\,d\eta}{|\sqrt{(p\xi\eta)^2\,d\xi^2+(p\eta\xi)^2\,d\eta^2+2\,(p\xi\eta)(p\eta\xi)\cos(\xi\eta)\,d\xi\,d\eta}|},$$

from Ex. 6, p. 42.

3. Reduce $(\nu x\overline{y}a)$.

$$(\nu xya)=\frac{d(x\overline{y}a)}{d\,xy}$$
$$=-\frac{|(ya)|\cos(\overline{ya}\,xy)(\tau xy)\,dx+|(xa)|\cos(\overline{xa}\,xy)(\tau yx)\,dy}{(\tau xy)\,dx+(\tau yx)\,dy},$$

/523/ from Ex. 5, p. 42

4. Shew that

$$(p\bar{x}\bar{y}a)^2 = \frac{(ay)^2(y\tau x)^2dx^2+(ax)^2(x\tau y)^2dy^2+2\,|\,(ax)(ay)\,|\cos(\overline{ax}\,\overline{ay})(y\tau x)(x\tau y)\,dx\,dy}{\{(y\tau x)\,dx+(x\tau y)\,dy\}^2}.$$

5. Reduce $(p\bar{x}\bar{y}a)$.

$$(p\bar{x}\bar{y}a) = -\sin(\bar{x}\bar{y}a)\left[\frac{d(x\bar{y}a)}{dxy}\right]_{a=\overline{xya}}$$

$$= -\sin(\bar{x}\bar{y}a)\,.\,\frac{-\,|(ya)|\cos(xy\,\overline{ya})(\tau xy)\,dx+|(xa)|\cos(\overline{yx}\,\overline{xa})(\tau yx)\,dy}{(\tau xy)\,dx+(\tau yx)\,dy}$$

$$a=\bar{x}\bar{y}a$$

$$= -\sin(\bar{x}\bar{y}a)\,.\,\frac{(xyay\bar{x}\bar{y}_{\frac{\pi}{2}})(\tau xy)\,dx+(\overline{xya}\,x\bar{x}\bar{y}_{\frac{\pi}{2}})(\tau yx)\,dy}{(\tau xy)\,dx+(\tau yx)\,dy}$$

$$= \frac{-(\bar{x}\bar{y}ay\bar{x}\bar{y}_{\frac{\pi}{2}})(\tau xy)\,dx-(xyaxxy_{\frac{\pi}{2}})(\tau yx)\,dy}{(\tau xy)\,dx+(\tau yx)\,dy}.$$

Hence
$$(p\bar{x}\bar{y}a)=\frac{(\tau xy)(ya)\,dx+(\tau yx)(xa)\,dy}{(\tau xy)\,dx+(\tau yx)\,dy}$$

6. Reduce $(p\bar{\xi}\bar{\eta}za)$.

$$(p\bar{\xi}\bar{\eta}za)=\frac{(za)(\tau\xi\eta z)\,d\xi\eta+(\xi\bar{\eta}a)(\xi\bar{\eta}\tau z)\,dz}{(\tau\xi\eta z)\,d\xi\eta+(\xi\eta\tau z)\,dz}$$

$$=\frac{-(za)\dfrac{(\eta z)(p\xi\eta)\,d\xi+(\xi z)(p\eta\xi)\,d\eta}{\sin^2(\xi\eta)}+(\xi\eta a)\,.\,\dfrac{(\xi\eta\tau z)\,dz}{\sin^2(\xi\eta)}}{-\dfrac{(\eta z)(p\xi\eta)\,d\xi+(\xi z)(p\eta\xi)\,d\eta}{\sin^2(\xi\eta)}+\dfrac{(\xi\eta\tau z)}{\sin(\xi\eta)}\,dz}$$

$$=\frac{(za)\{(\eta z)(p\xi\eta)\,d\xi+(\xi z)(p\eta\xi)\,d\eta\}-(\xi\eta a)(\xi\eta\tau z)\,dz}{(\eta z)(p\xi\eta)\,d\xi+(\xi z)(p\eta\xi)\,d\eta-\sin(\xi\eta)(\xi\eta\tau z)\,dz}$$

7. Reduce $(\tau x_{\acute\rho\acute\sigma\ \ \acute\omega}a)$.

$$(\tau x_{\acute\rho\acute\sigma\ \ \acute\omega}a)\,dx_{\acute\rho\acute\sigma\ \ \acute\omega}=(x_{\acute\rho\acute\sigma\ \ \acute\omega}a)^2\,dx_{\acute\rho\acute\sigma\ \ \acute\omega}\overline{a}$$

$$=\{(\tau xa)+\Sigma\hat\rho\sin(\rho\tau x)\}\,dx-\Sigma(xa\rho)\,d\hat\rho-\Sigma(xy\nu\rho)\,\hat\rho\,d\rho$$

$$+\Sigma\hat\rho^2d\rho+\sum_{\rho\neq\sigma}\hat\rho\hat\sigma(d\rho+d\sigma)\cos(\rho\sigma)+\Sigma(\hat\rho\,d\hat\sigma-\hat\sigma\,d\hat\rho)\sin(\rho\sigma),$$

from Ex. 2, p. 48.

8. Find the value of $(\tau x_{\rho\acute\sigma\ \ \acute\omega}a)$.

$$\sin(\tau x_{\acute\rho\acute\sigma\ \ \acute\omega}a)\,dx_{\acute\rho\acute\sigma\,.\ \acute\omega}=-d(x_{\acute\rho\acute\sigma\ \ \acute\omega}a)$$

$$=d\{-(xa)+\Sigma\hat\rho\sin(\rho a)\}$$

$$=\sin(\tau xa)\,dx+\Sigma\,d\hat\rho\sin(\rho a)-\Sigma\hat\rho\,d\rho\cos(\rho a)$$

9. Find the value of $(\nu x_{\acute\rho\acute\sigma\ldots\acute\phi\omega}a)$.

$$(\nu x_{\acute\rho\acute\sigma\ldots\acute\phi\omega}a)\,d\omega=d(x_{\acute\rho\acute\sigma\,.\ \phi\omega}a)$$

$$=d\{(ax\omega)+\Sigma\hat\rho\sin(\rho\omega)\}$$

$$=\sin(\tau x\omega)\,dx+(ax\nu\omega)\,d\omega+\Sigma\,d\hat\rho\sin(\rho\omega)-\Sigma\hat\rho\,d\rho\cos(\rho\omega)+d\omega\,\Sigma\hat\rho\cos(\rho\omega).$$

10. Find the value of $(p x_{\hat{\rho}\hat{\sigma}} .. \phi_\omega a)$.

$$(p x_{\hat{\rho}\hat{\sigma}} .. \phi_\omega a) = (x_{\hat{\rho}\hat{\sigma}} \quad \phi_\omega \, \nu x_{\hat{\rho}\hat{\sigma}} \quad \phi_\omega a)$$

$$= (\nu x_{\hat{\rho}\hat{\sigma}} \quad \phi_\omega \, \overline{a x_{\hat{\rho}\hat{\sigma}}} \quad \phi_\omega) \sin (a\omega)$$

$$= \sin (a\omega) \frac{\sin (\tau x \omega) \, dx + (a x \nu \omega) \, d\omega + \Sigma \, d\hat{\rho} \sin (\rho\omega) - \Sigma \, \hat{\rho} \, d\rho \cos (\rho\omega) + d\omega \, \Sigma \, \hat{\rho} \cos (\rho\omega)}{d\omega}$$

$$a = \overline{a x_{\hat{\rho}\hat{\sigma}} \ldots \phi_\omega}$$

$$= [\sin (\tau x\omega) \sin (a\omega) \, dx + (a x_{\hat{\rho}\hat{\sigma}} \ldots \phi_\omega x \nu \omega) \, d\omega + \sin (a\omega) \, \Sigma \, d\hat{\rho} \sin (\rho\omega)$$
$$- \sin (a\omega) \, \Sigma \, \hat{\rho} \, d\rho \cos (\rho\omega) + \sin (a\omega) \, d\omega \, \Sigma \, \hat{\rho} \cos (\rho\omega)]/d\omega$$

$$= \sin (\tau x\omega) \sin (a\omega) \frac{dx}{d\omega} + \sin (a\omega) \, \Sigma \frac{d\hat{\rho}}{d\omega} \sin (\rho\omega) - \sin (a\omega) \, \Sigma \, \hat{\rho} \frac{d\rho}{d\omega} \cos (\rho\omega)$$
$$+ (ax) - (x_{\hat{\rho}\hat{\sigma}} . \ \phi_\omega x) \cos (a\omega) + \sin (a\omega) \, \Sigma \, \hat{\rho} \cos (\rho\omega)$$

$$= \sin (\tau x\omega) \sin (a\omega) \frac{dx}{d\omega} + \sin (a\omega) \, \Sigma \frac{d\hat{\rho}}{d\omega} \sin (\rho\omega) - \sin (a\omega) \, \Sigma \, \hat{\rho} \frac{d\rho}{d\omega} \cos (\rho\omega)$$
$$+ (xa) + \Sigma \, \hat{\rho} \sin (a\rho).$$

11. Shew that

$$(p \, \Sigma a_r (x a_r) + a = 0 \ \beta) = [\Sigma_r a_r^2 da_r - \Sigma_{r \neq s} a_r a_s \{da_r (\nu a_r a_s \beta) + da_s (a_r \nu a_s \beta)\}$$
$$+ \Sigma_{r,s} (a_r a_s \beta)(a_r da_s - a_s da_r) + a \Sigma_r da_r \sin (a_r \beta) - da \Sigma_r a_r \sin (a_r \beta)]$$
$$\div [\Sigma a_r^2 da_r + \Sigma_{r \neq s} a_r a_s (da_r + da_s) \cos (a_r a_s) + \Sigma_{r,s} (a_r da_s - a_s da_r) \sin (a_r a_s)].$$

12. Shew that

$$(\tau \Sigma A_r (\xi a_r) + \Sigma B_r \cos (\xi \beta_r) = 0 \ c) \, d \{\Sigma A_r (\xi a_r) + \Sigma B_r \cos (\xi \beta_r) = 0\}$$
$$= - \Sigma_{r,s} (a_r a_s c)(A_r dA_s - A_s dA_r) + \Sigma_{r,s} dA_r B_s (c a_r \nu \beta_s)$$
$$- \Sigma A_r^2 da_r (\tau a_r c) + \Sigma_{r \neq s} A_r A_s \{da_r (a_s c \tau a_r) + da_s (a_r c \tau a_s)\}$$
$$+ \Sigma_{r,s} A_r B_s da_r \cos (\tau a_r \beta_s) + \Sigma_{r,s} A_s dB_r (a_s c \nu \beta_r) - \Sigma_{r,s} (B_r dB_s - B_s dB_r) \sin (\beta_r \beta_s)$$
$$- \Sigma_{r \neq s} B_r B_s (d\beta_r + d\beta_s) \cos (\beta_r \beta_s) - \Sigma_r B_r^2 d\beta_r - \Sigma_{r,s} A_s B_r d\beta_r (a_s c \beta_s).$$

13. Find the value of $\sin (\tau \xi \eta \tau \bar{\zeta} \omega)$.

$$\sin (\tau \xi \eta \tau \overline{\zeta \omega}) \, d \bar{\xi} \eta = - d (\tau \bar{\zeta} \omega \overline{\xi \eta})$$
$$= - d \frac{(\xi \eta \tau \bar{\zeta} \omega)}{\sin (\bar{\xi} \eta)},$$

$$\therefore \ \sin^2 (\xi \eta) \, d \overline{\xi \eta} . \sin (\tau \bar{\xi} \eta \tau \overline{\zeta \omega}) = - [\{(\nu \xi \eta \tau \overline{\zeta \omega}) \, d\xi + (\xi \nu \eta \tau \bar{\zeta \omega}) \, d\eta\} \sin (\xi \eta)$$
$$+ (\xi \eta \tau \overline{\zeta \omega}) \cos (\xi \eta) (d\xi - d\eta)]$$
$$= \sin (\tau \overline{\zeta \omega} \eta) [(p \xi \eta) \, d\xi - \sin (\tau \overline{\zeta \omega} \xi) (p \eta \xi) \, d\eta],$$

$$\therefore \ \sin^2 (\xi \eta) \, d \overline{\xi \eta} \sin^2 (\zeta \omega) \, d \overline{\zeta \omega} \sin (\tau \overline{\xi \eta} \tau \overline{\zeta \omega})$$
$$= - \sin^2 (\zeta \omega) [d (\overline{\zeta \omega} \eta)(p \xi \eta) \, d\xi - d (\bar{\zeta \omega} \xi)(p \eta \xi) \, d\eta]$$
$$= (p \xi \eta) \, d\xi \{(p \zeta \omega) \sin (\omega \eta) \, d\zeta + (p \omega \zeta) \sin (\zeta \eta) \, d\omega\}$$
$$- (p \eta \xi) \, d\eta \{(p \zeta \omega) \sin (\omega \xi) \, d\zeta + (p \omega \zeta) \sin (\zeta \xi) \, d\omega\}.$$

14. Find the value of $\sin(\tau x_{\rho}\tau y_{\sigma})$.

$$\sin(\tau x_{\rho}\tau y_{\sigma})\,dx_{\rho} = -d(\tau y_{\sigma} x_{\rho}) = +d\{-(\tau y_{\sigma} x) + \hat{\rho}\sin(\rho\tau y_{\sigma})\}$$
$$= \sin(\tau x\tau y_{\sigma})\,dx + d\hat{\rho}\sin(\rho\tau y_{\sigma}) - \hat{\rho}\,d\rho\cos(\rho\tau y_{\sigma}),$$
$$\therefore\ \sin(\tau x_{\rho}\tau y_{\sigma})\,dx_{\rho}dy_{\sigma} = dx\,d(y_{\sigma}\tau x) - d\hat{\rho}\,d(y_{\sigma}\rho) - \hat{\rho}\,d\rho\,d(y_{\sigma}\rho_{\frac{\pi}{2}})$$
$$= dx\,d\{(y\tau x) + \hat{\sigma}\sin(\sigma\tau x)\} - d\hat{\rho}\,d\{(y\rho) + \hat{\sigma}\sin(\sigma\rho)\}$$
$$- \hat{\rho}\,d\rho\,d\{(y\rho_{\frac{\pi}{2}}) + \hat{\sigma}\cos(\rho\sigma)\}$$
$$= dx\,dy\sin(\tau x\tau y) + dx\,d\hat{\sigma}\sin(\tau x\sigma) - dy\,d\hat{\rho}\sin(\tau y\rho)$$
$$+ dx\,\hat{\sigma}\,d\sigma\cos(\tau x\sigma) - y\,\hat{\rho}\,d\rho\cos(\tau y\rho)$$
$$+ (\hat{\rho}\,d\hat{\sigma} + \hat{\rho}\hat{\sigma}\,d\rho\,d\sigma)\sin(\rho\sigma) + (d\hat{\rho}\hat{\sigma}\,d\sigma - \hat{\rho}\,d\rho\,d\hat{\sigma})\cos(\rho\sigma).$$

§ 64. We may now differentiate determinates in a straightforward manner.

The formulae used are those of examples 1 and 5 of § 63, which are of a reciprocal character, viz.,

$$\sin^2(\xi\eta)\,d\overline{\xi\eta}\,(\tau\,\overline{\overline{\xi\eta}\,a}) = (p\xi\eta)(\eta a)\,d\xi + (p\eta\xi)(\xi a)\,d\eta \ldots\text{(A)},$$
$$(xy)^2\,d\overline{xy}\,(p\,\overline{\overline{xy}\,a}) = (\tau xy)(ya)\,dx + (\tau yx)(xa)\,dy \ldots\text{(B)}.$$

We consider determinates of six letters; this will be found to be quite general.

First, consider the line

$$\overline{\overline{\overline{\overline{\overline{xy}\,\zeta}\,a}\,\beta}\,c}.$$

Now $(xy\zeta a\beta c)^2\,d\,\overline{\overline{\overline{\overline{\overline{xy}\,\zeta}\,a}\,\beta}\,c}$

$$= (xy\zeta a\beta)^2\left[(\tau\,\overline{\overline{\overline{\overline{\overline{xy}\,\zeta}\,a}\,\beta}\,c})\,d\,\overline{\overline{\overline{\overline{xy}\,\zeta}\,a}\,\beta} + (\overline{\overline{\overline{\overline{xy}\,\zeta}\,a}\,\beta}\,\tau c)\,dc\right]$$
$$= (xy\zeta a\beta)^2(\tau\,\overline{\overline{\overline{\overline{\overline{xy}\,\zeta}\,a}\,\beta}\,c})\,d\,\overline{\overline{\overline{\overline{xy}\,\zeta}\,a}\,\beta} + (xy\zeta a\beta)(xy\zeta a\beta\tau c)\,dc$$
$$= (xy\zeta a)^2\left[(p\,\overline{\overline{\overline{\overline{xy}\,\zeta}\,a}\,\beta})(\beta c)\,d\,\overline{\overline{\overline{xy}\,\zeta}\,a} + (p\beta\,\overline{\overline{\overline{xy}\,\zeta}\,a})(\overline{\overline{\overline{xy}\,\zeta}\,a}\,c)\,d\beta\right]$$
$$+ (xy\zeta a\beta)(xy\zeta a\beta\tau c)\,dc \text{ by (A)},$$
$$= (\beta c)(xy\zeta a)^2(p\,\overline{\overline{\overline{\overline{xy}\,\zeta}\,a}\,\beta})\,d\,\overline{\overline{\overline{xy}\,\zeta}\,a} + (xy\zeta ap\beta)(xy\zeta ac)\,d\beta$$
$$+ (xy\zeta a\beta)(xy\zeta a\beta\tau c)\,dc$$
$$= (xy\zeta)^2(\beta c)\left[(\tau\,\overline{\overline{\overline{xy}\,\zeta}\,a})(a\beta)\,d\,\overline{\overline{xy}\,\zeta} + (\overline{\overline{xy}\,\zeta}\,\tau a)(\overline{\overline{xy}\,\zeta}\,\beta)\,da\right]$$
$$+ (xy\zeta ap\beta)(xy\zeta ac)\,d\beta + (xy\zeta a\beta)(xy\zeta a\beta\tau c)\,dc \text{ by (B)},$$
$$= (xy\zeta)^2(c\beta)(\beta a)(\tau\,\overline{\overline{xy}\,\zeta}\,a)\,d\,\overline{\overline{xy}\,\zeta} + (c\beta)(\beta\zeta yx)(\tau a\zeta yx)\,da$$
$$+ (p\beta a\zeta yx)(ca\zeta yx)\,d\beta + (xy\zeta a\beta)(xy\zeta a\beta\tau c)\,dc$$
$$= (xy)^2(c\beta)(\beta a)\left[(p\,\overline{xy}\,\zeta)(\zeta a)\,d\overline{xy} + (\overline{xy}\,p\zeta)(\overline{xy}\,a)\,d\zeta\right]$$
$$+ (c\beta)(\beta\zeta yx)(\tau a\zeta yx)\,da + (p\beta a\zeta yx)(ca\zeta yx)\,d\beta$$
$$+ (xy\zeta a\beta)(xy\zeta a\beta\tau c)\,dc \text{ by (A)},$$

$$\begin{aligned}
&= (c\beta)(\beta a)(a\zeta)(xy)^2(p\overline{xy}\zeta)\,d\overline{xy} + (c\beta)(\beta a)(ayx)(p\zeta yx)\,d\zeta \\
&\quad + (c\beta)(\beta\zeta yx)(\tau a\zeta yx)\,da + (p\beta a\zeta yx)(ca\zeta yx)\,d\beta \\
&\quad + (xy\zeta a\beta)(xy\zeta a\beta\tau c)\,dc \\
&= (c\beta)(\beta a)(a\zeta)[(\tau xy)(y\zeta)\,dx + (\tau yx)(x\zeta)\,dy] \\
&\quad + (c\beta)(\beta a)(ayx)(p\zeta yx)\,d\zeta + (c\beta)(\beta\zeta yx)(\tau a\zeta yx)\,da \\
&\quad + (p\beta a\zeta yx)(ca\zeta yx)\,d\beta + (xy\zeta a\beta)(xy\zeta a\beta\tau c)\,dc \text{ by (B)}, \\
&= (c\beta)(\beta a)(a\zeta)(\zeta y)(y\tau x)\,dx + (c\beta)(\beta a)(a\zeta)(\zeta x)(x\tau y)\,dy \\
&\quad + (c\beta)(\beta a)(ayx)(p\zeta yx)\,d\zeta + (c\beta)(\beta\zeta yx)(\tau a\zeta yx)\,da \\
&\quad + (p\beta a\zeta yx)(ca\zeta yx)\,d\beta + (xy\zeta a\beta)(xy\zeta a\beta\tau c)\,dc.
\end{aligned}$$

A rule is clearly discernible: take the letters in the reverse order of that in the determinate: viz. c, β, a, ζ, y, x.

To write down the coefficient of dx, replace x by τx, and we have c, β, a, ζ, y, τx.

Then the coefficient is the product of pairs in the same order.

The coefficient of dy is got by interchanging x and y.

The coefficient of $d\zeta$ is got by writing $p\zeta$ for ζ, the letters becoming

$$c,\ \beta,\ a,\ p\zeta,\ y,\ x.$$

We form pairs in order of all the letters before $p\zeta$; and afterwards with the letter before $p\zeta$ form the measure of all the letters succeeding $p\zeta$. The last component in the coefficient is the measure of $p\zeta$ with the letters following $p\zeta$. And similarly for the others; with the exception of the coefficient of dc which is easily written down.

It is easy to shew that the coefficients in a line determinate of odd number of letters as

$$(\xi\eta zab\gamma d)^2\, d\,\overline{\overline{\overline{\overline{\overline{\overline{\xi\eta}\,z}\,a}\,b}\,\gamma}\,d}$$

follow a similar rule.

Next, we consider a point determinate, namely,

$$\overline{\overline{\overline{\overline{\overline{\xi\eta}\,z}\,a}\,b}\,\gamma}$$

$$\begin{aligned}
&(\xi\eta zab\gamma)^4\,(d\,\overline{\overline{\overline{\overline{\overline{\xi\eta}\,z}\,a}\,b}\,\gamma})^2 \\
&= (\xi\eta zab)^4\,[(p\,\overline{\overline{\overline{\overline{\overline{\xi\eta}\,z}\,a}\,b}\,\gamma})^2(d\,\overline{\overline{\overline{\overline{\xi\eta}\,z}\,a}\,b})^2 + (\overline{\overline{\overline{\overline{\overline{\xi\eta}\,z}\,a}\,b}\,p\,\gamma})^2(d\gamma)^2 \\
&\quad + (p\,\overline{\overline{\overline{\overline{\overline{\xi\eta}\,z}\,a}\,b}\,\gamma})(\overline{\overline{\overline{\overline{\overline{\xi\eta}\,z}\,a}\,b}\,p\gamma})\cos(\overline{\overline{\overline{\overline{\overline{\xi\eta}\,z}\,a}\,b}\,\gamma})\,d\,\overline{\overline{\overline{\overline{\xi\eta}\,z}\,a}\,b}\,d\gamma].
\end{aligned}$$

This we can reduce if we reduce

$$(\xi\eta zab)^2\,(p\,\overline{\overline{\xi\eta}\,z\,\alpha\,b}\,\gamma)\,d\,\overline{\overline{\xi\eta}\,z\,\alpha\,b}.$$

This is equal to

$$(\xi\eta z\alpha)^2\,[(\tau\,\overline{\overline{\xi\eta}\,z\,\alpha\,b})\,(b\gamma)\,d\,\overline{\overline{\xi\eta}\,z\,\alpha}+(\overline{\overline{\xi\eta}\,z\,\alpha}\,\tau b)\,(\overline{\overline{\xi\eta}\,z\,\alpha}\,\gamma)\,db]$$

$$=(b\gamma)\,(\xi\eta z)^2\,[(p\,\overline{\overline{\xi\eta}\,z}\,\alpha)\,(\alpha b)\,d\,\overline{\overline{\xi\eta}\,z}+(\overline{\overline{\xi\eta}\,z}\,p\alpha)\,(\overline{\overline{\xi\eta}\,z}\,b)\,d\alpha]$$

$$+\,(\xi\eta z\alpha\tau b)\,(\xi\eta z\alpha\gamma)\,db$$

$$=(\gamma b)\,(b\alpha)\,\sin^2(\xi\eta)\,[(\tau\,\overline{\xi\eta}\,z)\,(z\alpha)\,d\,\overline{\xi\eta}+(\,\overline{\xi\eta}\,\tau z)\,(\,\overline{\xi\eta}\,\alpha)\,dz]$$

$$+\,(\gamma b)\,(bz\eta\xi)\,(p\alpha z\eta\xi)\,d\alpha+(\gamma\alpha z\eta\xi)\,(\tau b\alpha z\eta\xi)\,db$$

$$=(\gamma b)\,(b\alpha)\,(\alpha z)\,(z\eta)\,(\eta p\xi)\,d\xi+(\gamma b)\,(b\alpha)\,(\alpha z)(z\xi)\,(\xi p\eta)\,d\eta$$

$$+\,(\gamma b)\,(b\alpha)\,(\alpha\eta\xi)\,(\tau z\eta\xi)\,dz+(\gamma b)\,(bz\eta\xi)\,(p\alpha z\eta\xi)\,d\alpha$$

$$+\,(\gamma\alpha z\eta\xi)\,(\tau b\alpha z\eta\xi)\,db.$$

It is easy to see that the same rule holds good, as in the case of line determinates. Hence the differentials of determinates of simple elements may be written down.

§ 65. *Examples.*

1. Shew that

$$(xa_1\alpha_1a_2\ldots\alpha_{n-1}a_n)^2\,(p\,\overline{\overline{\overline{xa_1}\,\alpha_1\,a_2}\ldots\alpha_{n-1}\,a_n}\,\alpha_n)\,\frac{d}{dx}\,\overline{\overline{\overline{xa_1}\,\alpha_1\,a_2}\ldots\alpha_{n-1}\,a_n}$$

$$=(a_n\alpha_n)\,(\alpha_na_{n-1})\,(a_{n-1}\alpha_{n-1})\ldots(a_2\alpha_1)\,(\alpha_1a_1)\,(a_1\tau x).$$

2. Shew that

$$(\xi a_1\alpha_1a_2\ldots\alpha_{n-1}a_{n-1})^2\,(p\,\overline{\overline{\overline{\xi a_1}\,\alpha_1\,a_2}\ldots\alpha_{n-1}\,a_{n-1}}\,\alpha_n)\,\frac{d}{d\xi}\,\overline{\overline{\overline{\xi a_1}\,\alpha_1\,a_2}\ldots\alpha_{n-1}\,a_{n-1}}$$

$$=(\alpha_{n-1}a_{n-1})\,(a_{n-1}\alpha_{n-2})\ldots(a_2\alpha_1)\,(\alpha_1a_1)\,(a_1p\xi).$$

3. Shew that

$$(\xi a_1\alpha_1a_2\ldots\alpha_{n-1}a_n)^2\,(\tau\,\overline{\overline{\overline{\xi a_1}\,\alpha_1\,a_2}\ldots\alpha_{n-1}\,a_n}\,\alpha_n)\,\frac{d}{d\xi}\,\overline{\overline{\overline{\xi a_1}\,\alpha_1\,a_2}\ldots\alpha_{n-1}\,a_n}$$

$$=(\alpha_na_n)\,(a_n\alpha_{n-1})\ldots(a_2\alpha_1)\,(\alpha_1a_1)\,(a_1p\xi).$$

4. Shew that

$$(x\alpha_1a_1\alpha_2\ldots\alpha_{n-1}a_{n-1})^2\,(\tau\,\overline{\overline{\overline{x\alpha_1}\,a_1\,\alpha_2}\ldots\alpha_{n-1}\,a_{n-1}}\,\alpha_n)\,\frac{d}{dx}\,\overline{\overline{\overline{x\alpha_1}\,a_1\,\alpha_2}\ldots\alpha_{n-1}\,a_{n-1}}$$

$$=(a_n\alpha_n)\,(a_n\alpha_{n-1})\ldots(\alpha_2a_1)\,(a_1\alpha_1)\,(a_1\tau x).$$

5. Hence reduce

$$(x_1 x_2 \xi_1 x_3 \xi_2 \dots \xi_{n-1} x_{n+1})^2 \, (p \, \overline{\overline{x_1 x_2} \, \xi_1 x_3 \xi_2 \dots \xi_{n-1} \, x_{n+1} \xi_n}) \, d \, \overline{\overline{x_1 x_2} \, \xi_1 x_3 \dots \xi_{n-1} x_{n+1}}$$

$$(\xi_1 \xi_2 x_1 \xi_3 x_2 \dots \xi_n x_{n-1})^2 \, (p \, \overline{\overline{\xi_1 \xi_2} x_1 \xi_3 x_2 \dots \xi_n x_{n-1} \xi_{n+1}}) \, d \, \overline{\overline{\xi_1 \xi_2} \, x_1 \, \xi_3 \dots \xi_n x_{n-1}}$$

$$(\xi_1 \xi_2 x_1 \xi_3 x_2 \dots x_{n-1} \, \xi_{n+1})^2 \, (\tau \, \overline{\overline{\xi_1 \xi_2} \, x_1 \, \xi_3 \dots x_{n-1} \, \xi_{n+1} \, x_n}) \, d \, \overline{\overline{\xi_1 \xi_2} \, x_1 \, \xi_3 \dots x_{n-1} \, \xi_{n+1}}$$

$$(x_1 x_2 \xi_1 x_3 \xi_2 \dots x_n \xi_{n-1})^2 \, (\tau \, \overline{\overline{x_1 x_2} \, \xi_1 \, x_3 \dots x_n \, \xi_{n-1} \, x_{n+1}}) \, d \, \overline{\overline{x_1 x_2} \, \xi_1 \, x_3 \dots x_n \, \xi_{n-1}}.$$

6. Shew that

$$(x a_1 a_1 \dots a_{n-1} a_n)^2 \frac{d}{dx} \overline{\overline{x a_1} \, a_1 a_2 \dots a_{n-1} \, a_n} = (a_n a_{n-1}) \, (a_{n-1} \, a_{n-1}) \dots (a_1 a_1) \, (a_1 \tau x),$$

and $$(\xi a_1 a_1 \dots a_n a_n)^2 \frac{d}{d\xi} \overline{\overline{\xi a_1} \, a_1 \, a_2 \dots a_n \, a_n} = (a_n a_n) \, (a_n a_{n-1}) \dots (a_1 a_1) \, (a_1 p \xi).$$

7. Reduce $\dfrac{d}{dx} (\overline{\overline{x a_1} \, a_1 \dots a_{n-1} \, a_n \, b}).$

$$d \, (\overline{\overline{x a_1} \, a_1 \dots a_{n-1} \, a_n \, b}) = (\nu \, \overline{\overline{x a_1} \, a_1 \dots a_{n-1} \, a_n \, b}) \, d \, \overline{\overline{x a_1} \, a_1 \dots a_{n-1} \, a_n}$$

$$= (b \, p \, \overline{\overline{x a_1} \, a_1 \dots a_{n-1} \, a_n \, \overline{\overline{x a_1} \, a_1 \dots a_{n-1} \, a_{n \frac{\pi}{2}}}}) \, d \, \overline{\overline{x a_1} \, a_1 \dots a_{n-1} \, a_n}$$

$$= \{(b \, \overline{\overline{x a_1} \, a_1 \dots a_{n-1} \, a_{n \frac{\pi}{2}}}) - (p \, \overline{\overline{x a_1} \, a_1 \dots a_{n-1} \, a_n \, \overline{\overline{x a_1} \, a_1 \dots a_{n-1} \, a_{n \frac{\pi}{2}}}})\}$$

$$\times d \, \overline{\overline{x a_1} \, a_1 \dots a_{n-1} \, a_n}$$

$$\therefore \; (x a_1 a_1 \dots a_{n-1} \, a_n)^2 \frac{d}{dx} (\overline{\overline{x a_1} a_1 \dots a_{n-1} \, a_n \, b})$$

$$= (b \, \overline{\overline{x a_1} a_1 \dots a_{n-1} \, a_{n \frac{\pi}{2}}}) \, (a_n a_{n-1}) \, (a_{n-1} a_{n-1}) \dots (a_1 a_1) \, (a_1 \tau x)$$

$$- (\overline{\overline{x a_1} \, a_1 \dots a_{n-1} a_{n \frac{\pi}{2}} \, a_n}) \, (a_n a_{n-1}) \, (a_{n-1} a_{n-1}) \dots (a_1 a_1) \, (a_1 \tau x)$$

$$= (a_n \, a_{n-1}) \, (a_{n-1} \, a_{n-1}) \dots (a_1 \, a_1) \, (a_1 \tau x) \, |(b a_n)| \cos (\overline{b a_n \, \overline{\overline{x a_1} \, a_1 \dots a_{n-1} \, a_n}}).$$

8. Reduce $\dfrac{d}{dx} |(\overline{\overline{x a_1} \, a_1 a_2 \dots a_n \, a_n \, b})|.$

$$(x a_1 a_1 \dots a_n a_n)^2 \, d \, |(\overline{\overline{x a_1} \, a_1 \, a_2 \dots a_n \, a_n \, b})|$$

$$= -(x a_1 a_1 \dots a_n a_n)^2 \cos (\tau \, \overline{\overline{x a_1} \, a_1 \, a_2 \dots a_n \, a_n} \; \overline{\overline{x a_1} \, a_1 \, a_2 \dots a_n \, a_n \, b})$$

$$\times d \, \overline{\overline{x a_1} \, a_1 \, a_2 \dots a_n \, a_n}$$

$$= -\cos (\overline{\overline{x a_1} \, a_1 \, a_2 \dots a_n \, a_n \, b \, a_n}) \, (a_n a_n) \, (a_n \, a_{n-1}) \dots (a_2 a_1) \, (a_1 a_1) \, (a_1 \tau x) \, dx.$$

9. Reduce $\frac{d}{dx}(\overline{\overline{\overline{xa_1}\,a_1}\,a_2 \dots a_n\,a_n\,\beta})$.

$$(xa_1\,a_1\,a_2 \dots a_n\,a_n)^2 \frac{d}{dx}(\overline{\overline{\overline{xa_1}\,a_1}\,a_2 \dots a_n\,a_n\,\beta})$$

$$= -(xa_1\,a_1 \dots a_n\,a_n)^2 \sin(\tau\,\overline{\overline{\overline{xa_1}\,a_1}\,a_2 \dots a_n\,a_n\,\beta})\,d\,\overline{\overline{\overline{xa_1}\,a_1} \dots a_n\,a_n}/dx$$

$$= \sin(\beta a_n)\,(a_n a_n)\,(a_n a_{n-1}) \dots (a_2 a_1)\,(a_1 a_1)\,(a_1 \tau x).$$

10. Reduce $\frac{d}{d\xi}(\overline{\overline{\overline{\xi a_1}\,a_1}\,a_2 \dots a_{n-1}\,a_n\,\beta})$,

$$(\xi a_1\,a_1\,a_2 \dots a_{n-1}\,a_n)^2\,d\,(\overline{\overline{\overline{\xi a_1}\,a_1}\,a_2 \dots a_{n-1}\,a_n\,\beta})$$

$$= \sin(\beta a_n)\,(a_n a_{n-1})\,(a_{n-1} a_{n-1}) \dots (a_2 a_1)\,(a_1 \tau\,\overline{\xi a_1})\,d\,\overline{\xi a_1}\,\sin^2(\xi a_1)$$

$$= \sin(\beta a_n)\,(a_n a_{n-1})\,(a_{n-1} a_{n-1}) \dots (a_2 a_1)\,(a_1 a_1)\,(a_1 p \xi)\,d\xi.$$

11. Reduce $\frac{d}{d\xi}\,|\,(\overline{\overline{\overline{\xi a_1}\,a_1}\,a_2 \dots a_{n-1}\,a_n\,b})\,|$.

$$(\xi a_1\,a_1\,a_2 \dots a_{n-1}\,a_n)^2\,d\,|\,(\overline{\overline{\overline{\xi a_1}\,a_1}\,a_2 \dots a_{n-1}\,a_n\,b})\,|$$

$$= -\cos(\overline{\overline{\overline{\xi a_1}\,a_1}\,a_2 \dots a_{n-1}\,a_n\,b}\,a_n)\,(a_n a_{n-1})\,(a_{n-1} a_{n-1}) \dots (a_1 a_1)$$
$$(a_1 \tau\,\overline{\xi a_1})\,d\,\overline{\xi a_1}\,\sin^2(\xi a_1)$$

$$= -\cos(\overline{\overline{\overline{\xi a_1}\,a_1}\,a_2 \dots a_{n-1}\,a_n\,b}\,a_n)\,(a_n a_{n-1})\,(a_{n-1} a_{n-1}) \dots (a_1 a_1)\,(a_1 p\xi)\,d\xi.$$

12. Reduce $\frac{d}{d\xi}(\overline{\overline{\overline{\xi a_1}\,a_1}\,a_2 \dots a_n\,a_n\,b})$.

$$(\xi a_1\,a_1\,a_2 \dots a_n\,a_n)^2\,d\,(\overline{\overline{\overline{\xi a_1}\,a_1}\,a_2 \dots a_n\,a_n\,b})$$

$$= (a_n a_n)\,(a_n a_{n-1}) \dots (a_2 a_1)\,(a_1 a_1)\,|(b a_n)|\cos(\overline{\overline{ba_n}\,\overline{\overline{\xi a_1}\,a_1} \dots a_n\,a_n})$$
$$(a_1 \tau\,\overline{\xi a_1})\,d\,\overline{\xi a_1}\,\sin^2(\xi a_1)$$

$$= (a_n a_n)\,(a_n a_{n-1}) \dots (a_2 a_1)\,(a_1 a_1)\,|(b a_n)|\cos(\overline{\overline{ba_n}\,\overline{\overline{\xi a_1}\,a_1} \dots a_n\,a_n})\,(a_1 p\xi)\,d\xi.$$

CHAPTER IX

GENERALIZED DISPLACEMENT OF A POINT

§ 66. In the foregoing we have supposed that the consecutive position x' of a point x is given by

$$x' = x_{\tau x, \delta x}.$$

We shall now consider the most general case, viz.

$$x' = x_{\tau_1 x, \delta_1 x;\, \tau_2 x, \delta_2 x, \ldots \tau_n x, \delta_n x}.$$

§ 67. *To find dx.*

$$(\delta x)^2 = (x x_{\tau_1 x, \delta_1 x; \tau_2 x, \delta_2 x \ldots \tau_n x, \delta_n x})^2,$$

$$\therefore\ (dx)^2 = \Sigma (d_r x)^2 + 2\,\Sigma\, d_r x\, d_s x \cos (\tau_r x \tau_s x).$$

§ 68. *To find $d\tau x$.*

$$d\tau x = d\overline{xx}_{\tau_1 x, \delta_1 x; \tau_2 x, \delta_2 x \ldots \tau_n x, \delta_n x},$$

$$\therefore\ d\tau x\,(dx)^2 = \Sigma (d_r x)^2 d\tau_r x + \sum_{r \neq s} d_r x d_s x\,(d\tau_r x + d\tau_s x) \cos (\tau_r x \tau_s x)$$
$$+ \sum_{r,s} (d_r x d_s^2 x - d_s x d_r^2 x) \sin (\tau_r x \tau_s x),$$

from corollary, Ex. 2, § 55.

This gives $d\tau x$.

Defining ρx as $\dfrac{d\tau x}{dx}$ we have

$$\rho x = \{\Sigma (d_r x)^2 d\tau_r x + \sum_{r \neq s} d_r x d_s x\,(d\tau_r x + d\tau_s x) \cos (\tau_r x \tau_s x)$$
$$+ \sum_{r,s} (d_r x d_s^2 x - d_s x d_r^2 x) \sin (\tau_r x \tau_s x)\}$$
$$\div \{\Sigma (d_r x)^2 + 2 \sum_{r \neq s} d_r x d_s x \cos (\tau_r x \tau_s x)\}^{\frac{3}{2}}.$$

§ 69. *To find $(\nu x a)$.*

$$(\nu x a) = \frac{1}{2} \mathop{\mathrm{L}}_{x \to x'} \frac{(x'a)^2 - (xa)^2}{|(xx')|}$$
$$= \frac{1}{2} \mathop{\mathrm{L}}_{x \to x'} \frac{(x_{\tau_1 x, \delta_1 x; \tau_2 x, \delta_2 x \ldots \tau_n x, \delta_n x} a)^2 - (xa)^2}{|(xx')|}$$
$$= - \Sigma\, d_r x\, |(xa)| \cos (\overline{xa}\, \tau_r x)/dx,$$

$$\therefore\ (\nu x a)\, dx = - \Sigma\, d_r x\, (x a \nu_r x)$$

$$(\nu x a)\, dx = \Sigma\, (x_{\nu_r x} a)\, d_r x.$$

§ 70. *To find* $(\tau x a)$.

$$(\tau x a)\, dx = (xa)^2\, d\,\overline{xa} = (xa)^2\, (\overline{xa}\,\overline{x_{\tau_1 x, \delta_1 x \ldots \tau_n x, \delta_n x}\, a})$$

$$= |(xa)|\, (x\,\overline{x_{\tau_1 x, \delta_1 x \ldots \tau_n x, \delta_n x}\, a})$$

$$= (x_{\tau_1 x, \delta_1 x \ldots \tau_n x, \delta_n x}\,\overline{ax})\, |(xa)|,$$

$$\therefore\ (\tau x a)\, dx = |(ax)|\, \Sigma\, d_r x \sin(\overline{ax}\, \tau_r x)$$

$$= \Sigma\, d_r x\, (ax\, \tau_r x),$$

$$\therefore\ (\tau x a)\, dx = \Sigma\, d_r x\, (x_{\tau_r x}\, a).$$

Whence it is easy to see that any complex measure involving the generalized displacement can be reduced.

PLANE CURVES

CHAPTER X

REDUCTION OF COMPLEX FUNCTIONAL ELEMENTS

§ 71. We have the simple functional elements

$$\tau x,\ \nu x;\ p\xi,\ \nu\xi,$$

where x, ξ have successive positions.

Complex functional elements are as such,

$$p\tau x,\ \tau p\xi,\ \nu\tau x,\ \nu\nu x \text{ or } \nu^2 x,\ p\nu\xi.$$

For the reduction of these we have to make some initial assumptions.

Assuming
$$p\tau x = x,$$
$$\tau p\xi = \xi,$$

we may find the values of the others.

These define the geometry of plane curves.

For if x, x', x'' be three consecutive points on a curve

$$\tau x = \overline{xx'},\ (\tau x)' = \overline{x'x''},$$

$$\therefore\ p\tau x = \overline{\tau x\,(\tau x)'} = x.$$

Similarly for curves defined by a line variable.

We proceed next with the reduction of other complex elements.

We make the following definitions,

$$\frac{d\tau x}{dx} = \frac{d\nu x}{dx} = \rho x,$$

$$\frac{dp\xi}{d\xi} = \rho\xi,$$

so that $\rho x \,.\, \rho\tau x = 1$.

$\rho\xi$, $\dfrac{1}{\rho x}$ is called the radius of curvature of the curve at its line ξ or point x.

It is essential that the order along the curve in which its points take up successive positions, should be definite: this ensures a sense for τx.

§ 72. (i) We have $\nu\tau x = (p\tau x)_{\nu x} = x_{\nu x} = \nu x$.

(ii) $$(\nu^2 xa) = \frac{d(\nu xa)}{d\nu x} = \frac{d(x_{\nu x} a)}{d\nu x}$$

$$= \frac{d(ax\nu x)}{d\nu x} = \frac{\sin(\tau x \nu x)\, dx + (ax\nu^2 x)\, d\nu x}{d\nu x}$$

$$= \frac{1}{\rho x} - |(ax)| \sin(\overline{ax}\,\tau x)$$

$$= \frac{1}{\rho x} - (ax\tau x),$$

$$\therefore\ (\nu^2 xa) = \frac{1}{\rho x} - (\tau xa).$$

Corollary. $$(\nu^2 xx) = \frac{1}{\rho x}.$$

(iii) $$(p\nu x\alpha) = (\nu x \nu^2 x \alpha)$$

$$= (\nu^2 x\, \overline{\alpha\nu x}) \sin(\alpha\nu x)$$

$$= \left\{\frac{1}{\rho x} - (\tau x\, \overline{\alpha\nu x})\right\} \sin(\alpha\nu x)$$

$$= \frac{1}{\rho x} \cos(\alpha\tau x) + (\tau x \nu x\alpha),$$

$$\therefore\ (p\nu x\alpha) = (x\alpha) + \frac{\cos(\tau x\alpha)}{\rho x}.$$

Hence $$p\nu x = x_{\nu x,\, \frac{1}{\rho x}}.$$

(iv) $$\nu p\xi = \nu\tau p\xi = \nu\xi.$$

(v) $$(\nu^2\xi a) = \frac{d(\nu\xi a)}{d\nu\xi}$$

$$= \frac{d(ap\xi\nu\xi)}{d\nu\xi}$$

$$= \frac{\sin(\tau p\xi\nu\xi)\, dp\xi + (ap\xi\nu^2\xi)\, d\nu\xi}{d\nu\xi}$$

$$= \rho\xi - (ap\xi\xi),$$

$$\therefore\ (\nu^2\xi a) = \rho\xi - (\xi a).$$

(vi)
$$(p\nu\xi\alpha) = (\nu\xi\nu^2\xi\alpha) = (\nu^2\xi\overline{\alpha\nu\xi})\sin(\alpha\nu\xi)$$
$$= \{\rho\xi - (\xi\overline{\alpha\nu\xi})\}\sin(\alpha\nu\xi)$$
$$= \rho\xi\cos(\xi\alpha) + (\xi\nu\xi\alpha)$$
$$(p\nu\xi\alpha) = (p\xi\alpha) + \cos(\xi\alpha)\,\rho\xi.$$

§ 73. *Examples.*

1. Shew that

$$(\nu^{2n}\xi a) = \rho^{(2n-2)}\xi - \rho^{(2n-4)}\xi + \ldots + (-)^{n-1}\rho\xi + (-)^n(\xi a),$$
$$(\nu^{2n-1}\xi a) = \rho^{(2n-3)}\xi - \rho^{(2n-5)}\xi + \ldots + (-)^n\rho'\xi + (-)^{n-1}(\nu\xi a).$$

We have $\qquad (\nu^2\xi a) = \rho\xi - (\xi a),$

differentiating $\qquad (\nu^3\xi a) = -(\nu\xi a) + \rho'\xi,$

and $\qquad (\nu^4\xi a) = -(\nu^2\xi a) + \rho''\xi$

$$= -\rho\xi + \rho''\xi + (\xi a),$$
$$(\nu^6\xi a) = \rho\xi - \rho''\xi + \rho^{\text{IV}}\xi - (\xi a).$$

Hence the general formula.

Again $\qquad (\nu^5\xi a) = -(\nu^3\xi a) + \rho'''\xi$

$$= -\rho'\xi + \rho'''\xi + (\nu\xi a),$$

giving the other general formula.

2. Reduce $(p\nu^2\xi a)$.

$$(p\nu^2\xi a) = (p\nu\,.\,\nu\xi a) = (p\nu\xi a) + \cos(\nu\xi a)\,\rho\nu\xi$$
$$= (p\xi a) + \cos(\xi a)\,\rho\xi + \sin(\xi a)\,\rho'\xi.$$

3. Shew that $\qquad \rho\nu^n\xi = \rho^{(n)}\xi,$

$$(\nu^n\xi a) = (\xi a) - \frac{n\pi}{2}.$$

4. Shew that

$$(p\nu^n\xi a) - (p\nu^{n-1}\xi a) = \rho^{(n-1)}\xi\cos\left\{(n-1)\frac{\pi}{2} - (\xi a)\right\}.$$

Hence shew that

$$(p\nu^n\xi a) = (p\xi a) + \rho^{(n-1)}\xi\cos\left\{(n-1)\frac{\pi}{2} - (\xi a)\right\}$$
$$+ \rho^{(n-2)}\xi\cos\left\{(n-2)\frac{\pi}{2} - (\xi a)\right\}$$
$$+ \text{etc. etc.}$$

We have

$$(p\nu^n\xi a) = (p\nu\,.\,\nu^{n-1}\xi a) = \rho^{(n-1)}\xi\cos(\nu^{n-1}\xi a) + (p\nu^{n-1}\xi a)$$
$$= (p\nu^{n-1}\xi a) + \rho^{(n-1)}\xi\cos\left\{(n-1)\frac{\pi}{2} - (\xi a)\right\}.$$

5. Find the values of $(\nu^{2n-1}xa)$, $(\nu^{2n}xa)$.

$$(\nu^{2n}xa)=(\nu^{2n-1}.\nu xa)$$

$$=\left(\frac{d}{d\nu x}\right)^{2n-3}\rho\nu x-\left(\frac{d}{d\nu x}\right)^{2n-5}\rho\nu x+\ldots+(-)^n\left(\frac{d}{d\nu x}\right)\rho\nu x+(-)^{n-1}(\nu^2 xa)$$

$$=\left(\frac{1}{\rho x}\frac{d}{dx}\right)^{2n-3}\left(\frac{1}{\rho x}\right)-\left(\frac{1}{\rho x}\frac{d}{dx}\right)^{2n-5}\left(\frac{1}{\rho x}\right)-+\ldots$$

$$+(-)^n\left(\frac{1}{\rho x}\frac{d}{dx}\right)\left(\frac{1}{\rho x}\right)+(-)^n\frac{1}{\rho x}+(-)^n(\tau xa),$$

$$(\nu^{2n-1}xa)=(\nu^{2n-2}.\nu xa)$$

$$=\left(\frac{1}{\rho x}\frac{d}{dx}\right)^{2n-4}\left(\frac{1}{\rho x}\right)-\left(\frac{1}{\rho x}\frac{d}{dx}\right)^{2n-6}\left(\frac{1}{\rho x}\right)-+\ldots$$

$$+\frac{1}{\rho x}+(-)^{n-1}(\nu xa).$$

6. Find the value of $(p\nu^n xa)$.

$$(p\nu^n xa)=(p\nu^{n-1}\nu xa)=(p\nu xa)+\left(\frac{d}{d\nu x}\right)^{n-2}\rho\nu x\cos\left\{(n-2)\frac{\pi}{2}-(\nu xa)\right\}$$

$$+\left(\frac{d}{d\nu x}\right)^{n-3}\rho\nu x\cos\left\{(n-3)\frac{\pi}{2}-(\nu xa)\right\}+\ldots$$

$$=(xa)+\frac{\cos(\tau xa)}{\rho x}+\left(\frac{1}{\rho x}\frac{d}{dx}\right)^{n-2}\left(\frac{1}{\rho x}\right)\cos\left\{(n-1)\frac{\pi}{2}+(\tau xa)\right\}$$

$$+\left(\frac{1}{\rho x}\frac{d}{dx}\right)^{n-3}\left(\frac{1}{\rho x}\right)\cos\left\{(n-2)\frac{\pi}{2}+(\tau xa)\right\}+\ldots+\ldots.$$

CHAPTER XI

SUCCESSIVE DIFFERENTIATION OF MEASURES

§ 74. We propose to find the successive differentiations of the measures of two elements containing one variable element.

§ 75. (i) *Successive differentiation of* $\frac{1}{2}(xa)^2$.

$$\frac{1}{2}\frac{d}{dx}(xa)^2 = (\nu xa),$$

$$\frac{1}{2}\left(\frac{d}{dx}\right)^2(xa)^2 = \frac{d}{dx}(\nu xa) = \rho x\frac{d}{d\nu x}(\nu xa) = \rho x\,(\nu^2 xa)$$
$$= 1 - (\tau xa)\,\rho x,$$

$$\frac{1}{2}\left(\frac{d}{dx}\right)^3(xa)^2 = -\frac{d}{dx}\{(\tau xa)\,\rho x\}$$
$$= -(\rho x)^2(\nu xa) - \rho' x\,(\tau xa),$$

$$\frac{1}{2}\left(\frac{d}{dx}\right)^4(xa)^2 = -\frac{d}{dx}\{(\nu xa)(\rho x)^2 + (\tau xa)\rho' x\}$$
$$= -(\rho x)^3(\nu^2 xa) - 3\rho x\rho' x\,(\nu xa) - (\tau xa)\,\rho'' x$$
$$= -(\rho x)^2 + \{(\rho x)^3 - \rho'' x\}\,(\tau xa) - 3\rho x\rho' x\,(\nu xa),$$

$$\frac{1}{2}\left(\frac{d}{dx}\right)^5(xa)^2 = -2\rho x\rho' x + \rho x\,\{(\rho x)^3 - \rho'' x\}\,(\nu xa)$$
$$+\{3(\rho x)^2\rho' x - \rho''' x\}(\tau xa) + 3\rho x\rho' x\{1 + \rho x(\tau xa)\}$$
$$-3\,\{(\rho' x)^2 + \rho x\rho'' x\}\,(\nu xa)$$
$$= -5\rho x\rho' x + \{(\rho x)^4 - 4\rho x\rho'' x - 3\,(\rho' x)^2\}\,(\nu xa)$$
$$+\{6\,(\rho x)^2\,\rho' x - \rho''' x\}\,(\tau xa).$$

And generally, $\frac{1}{2}\left(\frac{d}{dx}\right)^n(xa)^2 = A_n(\tau xa) + B_n(\nu xa) + C_n$,

where A, B, C are polynomials of ρx, $\rho' x$, $\rho'' x\ldots$, and

$$A_{n+1} = A_n' - B_n\rho x,$$
$$B_{n+1} = B_n' + A_n\rho x,$$
$$C_{n+1} = C_n' + B_n.$$

(ii) *Successive differentiation of* $|(xa)|$.

$$\frac{d}{dx}|(xa)| = \frac{(\nu xa)}{|(xa)|},$$

$$\left(\frac{d}{dx}\right)^2 |(xa)| = \frac{d}{dx}\left\{\frac{(\nu xa)}{|(xa)|}\right\} = \frac{\rho x\,(\nu^2 xa)}{|(xa)|} - \frac{(\nu xa)^2}{|(xa)|^3}$$

$$= \frac{1-\rho x\,(\tau xa)}{|(xa)|} - \frac{(\nu xa)^2}{|(xa)|^3},$$

$$\left(\frac{d}{dx}\right)^3 |(xa)| = -\frac{\rho' x\,(\tau xa)}{|(xa)|} - \frac{(\rho x)^2(\nu xa)}{|(xa)|} - \frac{3\,(\nu xa)}{|(xa)|^3}$$

$$+ \frac{3\rho x\,(\tau xa)\,(\nu xa)}{|(xa)|^3} + \frac{3\,(\nu xa)^3}{|(xa)|^5},$$

$$\left(\frac{d}{dx}\right)^4 |(xa)| = \frac{(\rho x)^2}{|(xa)|} - \frac{3}{|(xa)|^3} + (\tau xa)\left\{-\frac{\rho'' x}{|(xa)|} + \frac{(\rho x)^3}{|(xa)|} - \frac{6\rho x}{|(xa)|^3}\right\}$$

$$-(\nu xa)\frac{3\rho x\rho' x}{|(xa)|} + (\tau xa)(\nu xa)\frac{4\rho' x}{|(xa)|^3}$$

$$-(\tau xa)^2\frac{3\,(\rho x)^2}{|(xa)|^3} + (\nu xa)^2\left\{\frac{4\,(\rho x)^2}{|(xa)|^3} - \frac{18}{|(xa)|^5}\right\}$$

$$-(\tau xa)(\nu xa)^2\frac{18\rho x}{|(xa)|^5} - \frac{15\,(\nu xa)^4}{|(xa)|^7}$$

(iii) *Successive differentiation of* $(x\alpha)$.

$$\frac{d\,(x\alpha)}{dx} = -\sin(\tau x\alpha),$$

$$\left(\frac{d}{dx}\right)^2 (x\alpha) = -\frac{d}{dx}\{\sin(\tau x\alpha)\} = \rho x\cos(\tau x\alpha),$$

$$\left(\frac{d}{dx}\right)^3 (x\alpha) = \frac{d}{dx}\{\rho x\cos(\tau x\alpha)\}$$

$$= \rho' x\cos(\tau x\alpha) + (\rho x)^2\sin(\tau x\alpha),$$

$$\left(\frac{d}{dx}\right)^4 (x\alpha) = \rho'' x\cos(\tau x\alpha) + 3\rho' x\rho x\sin(\tau x\alpha) - (\rho x)^3\cos(\tau x\alpha)$$

$$= \{\rho'' x - (\rho x)^3\}\cos(\tau x\alpha) + 3\rho x\rho' x\sin(\tau x\alpha),$$

$$\left(\frac{d}{dx}\right)^5 (x\alpha) = -\frac{d}{dx}\{(\rho x)^3 - \rho'' x\}\cos(\tau x\alpha) - \{(\rho x)^4 - \rho x\rho'' x\}\sin(\tau x\alpha)$$

$$+ 3\frac{d}{dx}(\rho x\rho' x)\sin(\tau x\alpha) - 3\,(\rho x)^2\rho' x\cos(\tau x\alpha)$$

$$= \{6\,(\rho x)^2\rho' x - \rho''' x\}\cos(\tau x\alpha)$$

$$+ \sin(\tau x\alpha)\{4\rho x\rho'' x + 3\,(\rho' x)^2 - (\rho x)^4\}.$$

And generally, $\left(\dfrac{d}{dx}\right)^n (x\alpha) = A_n\sin(\tau x\alpha) + B_n\cos(\tau x\alpha)$,

where A, B are polynomials in ρx, $\rho' x$, $\rho'' x\ldots$ and

$$A_{n+1} = A_n' + B_n\rho x,\quad B_{n+1} = B_n' - A_n\rho x.$$

(iv) *Successive differentiation of* (ξa).

$$\frac{d(\xi a)}{d\xi} = (\nu\xi a),$$

$$\left(\frac{d}{d\xi}\right)^2 (\xi a) = \frac{d}{d\xi}(\nu\xi a) = (\nu^2\xi a),$$

and generally,

$$\left(\frac{d}{d\xi}\right)^n (\xi a) = (\nu^n\xi a),$$

$$\therefore \left(\frac{d}{d\xi}\right)^{2n} (\xi a) = \rho^{(2n-2)}\xi - \rho^{(2n-4)}\xi + \ldots + (-)^{n-1}\rho\xi + (-)^n(\xi a),$$

$$\left(\frac{d}{d\xi}\right)^{2n-1} (\xi a) = \rho^{(2n-3)}\xi - \rho^{(2n-5)}\xi + \ldots + (-)^n\rho'\xi + (-)^{n-1}(\nu\xi a).$$

(v) *Successive differentiation of* $(\xi\alpha)$.

$$\frac{d}{d\xi}(\xi\alpha) = -1.$$

Hence $$\left(\frac{d}{d\xi}\right)^n (\xi\alpha) = 0, \quad n > 1.$$

Thus we see that we have general expressions for the nth differentiations of the measures of two elements in which the variable element is a line. In some cases this renders the line more useful as a variable than the point.

It will be seen in the next chapter that both (ξa) and $F\{(\xi\alpha)\}$ can also be integrated in known quantities.

§ 76. *Examples.*

1. Find $$\left(\frac{d}{dx}\right)^n (x\overline{a}\,\beta), \quad n = 1, 2, 3.$$

$$\frac{d}{dx}(\overline{xa}\,\beta) = -\frac{(\tau xa)}{(xa)^2},$$

$$\left(\frac{d}{dx}\right)^2 (\overline{xa}\,\beta) = -\frac{d}{dx}\frac{(\tau xa)}{(xa)^2} = -\frac{\rho x(\nu xa)}{(xa)^2} + \frac{2(\tau xa)(\nu xa)}{(xa)^4},$$

$$\left(\frac{d}{dx}\right)^3 (\overline{xa}\,\beta) = -\frac{\rho' x(\nu xa)}{(xa)^2} + \frac{\rho x\{1 - \rho x(\tau xa)\}}{(xa)^2} + \frac{2\rho x(\nu xa)^2}{(xa)^4}$$
$$+ \frac{2\rho x(\nu xa)^2}{(xa)^4} + \frac{2(\tau xa)\{1 - \rho x(\tau xa)\}}{(xa)^4} + \frac{8(\tau xa)(\nu xa)^2}{(xa)^6}$$
$$= \frac{\rho x}{(xa)^2} - (\tau xa)\left\{\frac{\rho x^2}{(xa)^2} - \frac{2}{(xa)^4}\right\} - (\nu xa)\frac{\rho' x}{(xa)^2}$$
$$+ \frac{4\rho x(\nu xa)^2}{(xa)^4} - \frac{2\rho x(\tau xa)^2}{(xa)^4} + \frac{8(\tau xa)(\nu xa)^2}{(xa)^6}.$$

2. Find $$\left(\frac{d}{d\xi}\right)^n (\overline{\xi a}\beta),\quad n=1, 2, 3.$$

We have $$\frac{d}{d\xi}(\overline{\xi a}\beta) = -\frac{(p\xi a)\sin(a\beta)}{\sin^2(\xi a)},$$

$$\left(\frac{d}{d\xi}\right)^2 (\overline{\xi a}\beta) = -\sin(a\beta)\frac{d}{d\xi}\frac{(p\xi a)}{\sin^2(\xi a)}$$

$$= +\sin(a\beta)\,.\,\frac{\sin^2(\xi a) - (p\xi a)\,.\,2\cos(\xi a)}{\sin^3(\xi a)}$$

$$= \sin(a\beta)\,.\,\frac{\sin^2(\xi a) - 2(p\xi a)\cos(\xi a)}{\sin^3(\xi a)},$$

$$\left(\frac{d}{d\xi}\right)^3 (\overline{\xi a}\beta) = \sin(a\beta)\,.\,\{4\sin(\xi a)\cos(\xi a) - 2(p\xi a)\sin(\xi a)\}\sin(\xi a)$$

$$+\frac{\{\sin^2(\xi a) - 2(p\xi a)\cos(\xi a)\}\,.\,3\cos(\xi a)}{\sin^4(\xi a)}$$

$$= \frac{\sin(a\beta)}{\sin^4(\xi a)}\left[7\sin^2(\xi a)\cos(\xi a) - 2(p\xi a)\{1 + 2\cos^2(\xi a)\}\right].$$

CHAPTER XII

INTEGRATION OF MEASURES

§ 77. We define integration as the inverse process of differentiation. Thus, for example,

$$\frac{d}{d\xi}(\xi a) = (\nu\xi a).$$

We have with the usual symbol of integration,

$$\int (\nu\xi a)\, d\xi = (\xi a).$$

If C be a constant

$$\int (\nu\xi a)\, d\xi = (\xi a) + C.$$

§ 78. Integration may as usual be defined as the limit of the sum of a series. For the integrand being an algebraic quantity we must have, if $f(\xi)$ be a function of absolutes containing ξ, that

$$\int_\alpha^\beta f(\xi)\, d\xi = \underset{n \to \infty}{L} \{f(\alpha) + f(\alpha_1) + \dots + f(\alpha_n \text{ or } \beta)\} h,$$

where $h = (\alpha\alpha_1) = (\alpha_1\alpha_2) = \dots = (\alpha_{n-1}\alpha_n) = \dfrac{(\alpha\beta)}{n}$; α, β are the limits between which ξ takes all values.

§ 79. In the foregoing it is generally necessary to suppose that ξ envelopes a curve, and so also when we consider the integration of measures of a variable point, we have in general to suppose that x traces a curve.

We have as before

$$\frac{d}{dx}\int f(x)\, dx = f(x),$$

and $$\int_a^b f(x)\, dx = \underset{n \to \infty}{L} \{f(a) + f(a_1) + \dots + f(a_n \text{ or } b)\} h,$$

where $$h = |(aa_1)| = |(a_1a_2)| = \ldots = |(a_{n-1}a_n)|$$
$$= \frac{\text{length of curve from } a \text{ to } b}{n}.$$

§ 80. *Integrals of measures of elements containing a point variable:*

(1) $\int_m^n dx =$ length of curve from m to n.

For the indefinite integral we shall write

$$\int dx = \text{lin}\, x.$$

(2) $\int_m^n (\tau xa)\, dx = (amn) +$ the area of space between $\overline{mn}$ and the curve from m to n.

For the indefinite integral we shall write

$$\int (\tau xa)\, dx = \text{seg}\, x.$$

(3) $$\int (\nu xa)\, dx = \tfrac{1}{2}(xa)^2.$$

From which we have

(4) $\int (\nu xa)\, \phi\, |(xa)|\, dx = \int y\, \phi(y)\, dy,$ where $y = |(xa)|$.

For $$\frac{d}{dx}\int y\,\phi(y)\,dy = \frac{d\,|(xa)|}{dx}\,\frac{d}{dy}\int y\,\phi(y)\,dy$$
$$= \frac{(\nu xa)}{|(xa)|}\cdot|(xa)|\,\phi\,|(xa)|.$$

(5) $\int \sin(\tau x\alpha)\, \phi(x\alpha)\, dx = -\int \phi(y)\, dy,$ where $y = (x\alpha)$.

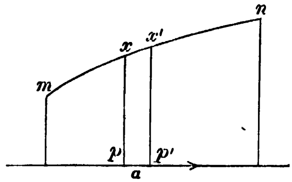
For $\dfrac{d}{dx}\displaystyle\int \phi(y)\, dy = \dfrac{d(x\alpha)}{dx}\,\dfrac{d}{dy}\displaystyle\int \phi(y)\, dy = -\sin(\tau x\alpha)\,\phi(x\alpha).$

(6) $\int \dfrac{(\tau xa)}{(xa)^2}\,\phi(\overline{xa}\beta)\, dx = -\int \phi(y)\, dy,$ where $y = (\overline{xa}\beta)$.

For $\dfrac{d}{dx}\displaystyle\int \phi(y)\, dy = \dfrac{d(\overline{xa}\beta)}{dx}\,\dfrac{d}{dy}\displaystyle\int \phi(y)\, dy = -\dfrac{d\overline{xa}}{dx}\,\phi(y).$

(7) $\int_m^n (x\alpha)\cos(\tau x\alpha)\, dx =$ trapezium $(mn\alpha) +$ seg x.

This we prove by the summation process. Let m, n be two points on a curve. From points on the curve draw ordinates to α. Let x be a point on the curve, x' a consecutive position. Let p, p' be the feet of the ordinates.



Then it is easy to see that, since

$$dx \cos(\tau x\alpha) = |(pp')|,$$

$$\int_m^n (x\alpha)\cos(\tau x\alpha)\,dx = \text{trapezium } (mn\alpha)$$
$$+ \text{area between } (mn) \text{ and the curve.}$$

Hence the indefinite integral is

$$\int (x\alpha)\cos(\tau x\alpha)\,dx = \text{seg } x.$$

(8) From (6) we have

$$\int \frac{(\tau xa)}{(xa)^2}\sin(\overline{xa}\beta)\,dx = -\int \sin y\,dy = \cos y = \cos(\overline{xa}\beta),$$

$$\therefore \int \frac{(\tau xa)}{|(xa)|^3}\{(a\beta)-(x\beta)\}\,dx = \cos(\overline{xa}\beta).$$

If $(a\beta)=0$, we may put $\beta = \overline{ab}$, and we have

$$\int \frac{(\tau xa)(xab)}{|(xa)|^3}\,dx = -|(ab)|\cos(\overline{xa}\,ab).$$

(9) Again from (6) we have

$$\int \frac{(\tau xa)}{(xa)^2}\,\frac{1}{\sin(\overline{xa}\beta)}\,dx = -\int (\sin y)^{-1}\,dy$$
$$= -\log\tan\tfrac{1}{2}y,$$

$$\therefore \int \frac{(\tau xa)\,dx}{|(xa)|\,\{(a\beta)-(x\beta)\}} = -\log\tan\tfrac{1}{2}(\overline{xa}\beta).$$

From which we have

$$\int \frac{(\tau xa)\,dx}{|(xa)|\,(xab)} = \frac{1}{|(ab)|}\log\tan\tfrac{1}{2}(\overline{xa}\,\overline{ab}).$$

(10) We have also from (6)

$$\int \frac{(\tau xa)}{(xa)^2}\cdot\frac{dx}{\sin^2(\overline{xa}\beta)} = -\int \operatorname{cosec}^2 y\,dy = \cot y = \cot(\overline{xa}\beta),$$

from which
$$\int \frac{(\tau xa)\,dx}{\{(a\beta)-(x\beta)\}^2} = \cot(\overline{xa}\beta),$$

and therefore
$$\int \frac{(\tau xa)\,dx}{(xab)^2} = \frac{1}{(ab)^2}\cot(\overline{xa}\,\overline{ab}).$$

(11) $\int \{(\tau xa) - (\tau xb)\} \phi(xab)\, dx = \int \phi(y)\, dy$, where $y = (xa$

For $\dfrac{d}{dx}\int \phi(y)\, dy = \dfrac{d(xab)}{dx}\dfrac{d}{dy}\int \phi(y)\, dy = \{(\tau xa) - (\tau xb)\}\,\phi(xab)$.

§ 81. *Integrals of measures of elements containing a line variable:*

(1) $\int d\xi = -(\xi\alpha)$, where α is a fixed line.

(2) $\int \phi(\xi\alpha)\, d\xi = -\int \phi(y)\, dy$, where $y = (\xi\alpha)$.

(3) $\int (\xi a)\, d\xi = \operatorname{lin} p\xi - (\nu\xi a)$.

For differentiating R.H.S. we have

$$\rho\xi - (\nu^2\xi a) = (\xi a).$$

(4) $\int (\nu\xi a)\, \phi(\xi a)\, d\xi = \int \phi(y)\, dy$, where $y = (\xi a)$.

(5) $\int \dfrac{(p\xi\alpha)}{\sin^2(\xi\alpha)}\phi(\bar{\xi\alpha}\beta)\, d\xi = -\dfrac{1}{\sin(\bar{\alpha\beta})}\int \phi(y)dy$,

where $y = (\bar{\xi\alpha}\beta)$.

(6) In (5) put $\phi(y) = \dfrac{1}{y^2}$, then

$$\int \frac{(p\xi\alpha)}{\sin^2(\xi\alpha)}\frac{\sin^2(\xi\alpha)}{(\xi\alpha\beta)^2}\, d\xi = -\frac{1}{\sin(\alpha\beta)}\int \frac{dy}{y^2} = \frac{1}{\sin(\alpha\beta)}\frac{\sin(\xi\alpha)}{(\xi\alpha\beta)},$$

$$\therefore \int \frac{(p\xi\alpha)}{(\xi\alpha\beta)^2}\, d\xi = \frac{1}{\sin(\alpha\beta)(\bar{\xi\alpha}\beta)}.$$

(7) In (5) put $\phi(y) = y$, then

$$\int \frac{(p\xi\alpha)(\xi\alpha\beta)}{\sin^3(\xi\alpha)}\, d\xi = -\frac{1}{2}\frac{(\bar{\xi\alpha}\beta)^2}{\sin(\alpha\beta)}.$$

(8) In (5) put $\phi(y) = \dfrac{1}{y}$, then

$$\int \frac{(p\xi\alpha)\, d\xi}{\sin(\xi\alpha)(\xi\alpha\beta)} = -\frac{1}{\sin(\alpha\beta)}\log(\bar{\xi\alpha}\beta).$$

(9) $\int \dfrac{(p\xi\alpha)\cos(\xi\alpha)}{\sin^2(\xi\alpha)}\, d\xi = \dfrac{(p\xi\alpha)}{\sin(\xi\alpha)} + \operatorname{lin} p\xi$.

For differentiating R.H.S. we have

$$\rho\xi - \frac{\rho\xi \sin(\xi\alpha)}{\sin(\xi\alpha)} + \frac{(p\xi\alpha)\cos(\xi\alpha)}{\sin^2(\xi\alpha)} = \frac{(p\xi\alpha)\cos(\xi\alpha)}{\sin^2(\xi\alpha)}.$$

$$(10) \quad \int \frac{(p\xi\alpha)^2}{\sin^2(\xi\alpha)}\, d\xi = -2 \operatorname{seg} p\xi - (p\xi\alpha)^2 \cot(\xi\alpha).$$

§ 82. Integrals of point elements may be converted into integrals of line elements by the substitution $x = p\xi$, and integrals of line elements into integrals of point elements by the substitution $\xi = \tau x$.

§ 83. A function of measures of a variable point is differentiated along the normal and integrated along the tangent of a curve. If the integral be such that it is independent of the path of integration, the function satisfies Laplace's equation. For let the function of measures be reduced to a function of measures of the variable point x and two fixed lines α, β at right angles, thus

$$F\{(x\alpha), (x\beta)\}.$$

Then the integral is

$$-\int \left\{\frac{\partial F}{\partial(x\alpha)} \sin(\nu x\alpha) + \frac{\partial F}{\partial(x\beta)} \sin(\nu x\beta)\right\} dx.$$

Since this is to be independent of the path of integration it must be equal to a function of $(x\alpha)$, $(x\beta)$, say $\phi\{(x\alpha), (x\beta)\}$,

$$\therefore \frac{\partial F}{\partial(x\alpha)} \sin(\nu x\alpha) + \frac{\partial F}{\partial(x\beta)} \sin(\nu x\beta)$$

$$= \frac{\partial \phi}{\partial(x\alpha)} \sin(\tau x\alpha) + \frac{\partial \phi}{\partial(x\beta)} \sin(\tau x\beta),$$

$$\therefore -\frac{\partial F}{\partial(x\alpha)} \sin(\tau x\beta) + \frac{\partial F}{\partial(x\beta)} \sin(\tau x\alpha)$$

$$= \frac{\partial \phi}{\partial(x\alpha)} \sin(\tau x\alpha) + \frac{\partial \phi}{\partial(x\beta)} \sin(\tau x\beta).$$

$$\therefore \frac{\partial F}{\partial(x\beta)} = \frac{\partial \phi}{\partial(x\alpha)}; \quad \frac{\partial F}{\partial(x\alpha)} = -\frac{\partial \phi}{\partial(x\beta)},$$

$$\therefore \frac{\partial^2 F}{\partial(x\alpha)^2} + \frac{\partial^2 F}{\partial(x\beta)^2} = 0.$$

MISCELLANEOUS EXAMPLES.

1. A curve is given by the general equation

$$f\{|(xa)|,\ |(xb)| \dots (xa),\ (x\beta) \dots\} = 0,$$

to find the radius of curvature at the point x.

Differentiating once

$$\Sigma \frac{\partial f}{\partial\,|(xa)|} \frac{d\,|(xa)|}{dx} + \Sigma \frac{\partial f}{\partial\,(xa)} \frac{d\,(xa)}{dx} = 0.$$

$$\therefore\ \Sigma \frac{\partial f}{\partial\,|(xa)|} \frac{(\nu xa)}{|(xa)|} - \Sigma \frac{\partial f}{\partial\,(xa)} \sin(\tau xa) = 0 \quad \dots\dots\dots\dots\text{(i)}.$$

Again, differentiating

$$\Sigma \frac{\partial^2 f}{\partial\,|(xa)|^2} \frac{(\nu xa)^2}{(xa)^2} - \Sigma \frac{\partial f}{\partial\,|(xa)|} \left\{\frac{1}{|(xa)|} - \frac{\rho x\,(\tau xa)}{|(xa)|} - \frac{(\nu xa)^2}{(xa)^3}\right\}$$
$$+ \Sigma \frac{\partial^2 f}{\partial\,(xa)^2} \sin^2(\tau xa) + \Sigma \frac{\partial f}{\partial\,(xa)} \cos(\tau xa)\,\rho x = 0,$$

$$\therefore\ -\rho x \left\{\Sigma \frac{\partial f}{\partial\,(xa)} \cos(\tau xa) + \Sigma \frac{\partial f}{\partial\,|(xa)|} \frac{(\tau xa)}{|(xa)|}\right\}$$
$$= \Sigma \frac{\partial^2 f}{\partial\,|(xa)|^2} \frac{(\nu xa)^2}{(xa)^2} - \Sigma \frac{\partial f}{\partial\,|(xa)|} \left\{\frac{1}{|(xa)|} - \frac{(\nu xa)^2}{(xa)^3}\right\} + \Sigma \frac{\partial^2 f}{\partial\,(xa)^2} \sin^2(\tau xa) \dots\text{(ii)}.$$

Eliminating τx, νx from (i) and (ii) we get ρx.

2. Find the radius of curvature at a point of a curve given by bi-radial co-ordinates.

Let a, b be the points of reference.

Let x be a point of the curve.

Let $$r = |(xa)|,\ s = |(xb)|.$$

Let $$|(ab)| = c.$$

To find ρx in terms of r, s.

$$\frac{dr}{dx} = \frac{(\nu xa)}{r},\ \frac{ds}{dx} = \frac{(\nu xb)}{s},$$

$$\therefore\ \frac{r\,dr}{s\,ds} = \frac{(\nu xa)}{(\nu xb)} \quad \dots \dots\dots\dots\dots\dots\text{(i)},$$

$$\therefore\ \frac{1}{r}\frac{dr}{dx} - \frac{1}{s}\frac{ds}{dx} + \frac{ds}{dr}\frac{d^2 r}{ds\,dx} = \left\{\frac{(\nu^2 xa)}{(\nu xa)} - \frac{(\nu^2 xb)}{(\nu xb)}\right\}\rho x$$
$$= \left\{\frac{1 - (\tau xa)\,\rho x}{(\nu xa)} - \frac{1 - (\tau xb)\,\rho x}{(\nu xb)}\right\}$$
$$= \frac{(\nu xb) - (\nu xa)}{(\nu xa)(\nu xb)} + \left\{\frac{(\tau xb)}{(\nu xb)} - \frac{(\tau xa)}{(\nu xa)}\right\}\rho x$$
$$= \frac{(\nu xb) - (\nu xa)}{(\nu xa)(\nu xb)} - \frac{(\tau x \nu x ab)}{(\nu xa)(\nu xb)}\,\rho x,$$

$$\therefore\ (xab)\,\rho x = (\nu xb) - (\nu xa) - \left\{\frac{1}{r}\frac{dr}{dx} - \frac{1}{s}\frac{ds}{dx} + \frac{ds}{dr}\frac{d^2 r}{ds\,dx}\right\}(\nu xa)(\nu xb).$$

From a, b, x, νx we have the eliminant

$$(abx)^2=(a\nu x)^2(bx)^2+(b\nu x)^2(ax)^2-\{(ax)^2+(bx)^2-(ab)^2\}(a\nu x)(b\nu x).$$

From (i) we have

$$(\nu \dot{x} a)=rdr/dx,\ (\nu x b)=sds/dx.$$

Let $$(\overline{bx}\,\overline{xa})=\theta.$$

Then $$(dx)^2(abx)^2=r^2s^2(dr^2+ds^2+2dr\,ds\cos\theta),$$

$$\therefore\ (dx)^2=\frac{r^2s^2}{(abx)^2}(dr^2+ds^2+2dr\,ds\cos\theta),$$

$$\therefore\ \rho x=\frac{1}{s}\frac{dr}{d\sigma}-\frac{1}{r}\frac{ds}{d\sigma}-\frac{\Delta^2}{r^2s^2}\frac{dr}{d\sigma}\frac{ds}{d\sigma}\left\{\frac{dr}{rd\sigma}-\frac{ds}{sd\sigma}+\frac{ds}{dr}\frac{ds}{d\sigma}\frac{d^2r}{ds^2}\right\},$$

where $$4\Delta^2=2r^2s^2+2c^2(r^2+s^2)-r^4-s^4-c^4,$$

$$d\sigma^2=dr^2+ds^2+2dr\,ds\cos\theta,$$

$$\cos\theta=\frac{c^2-r^2-s^2}{2rs}.$$

3. Rectify the curve, whose tangential equation is

$$\Sigma A_r(\xi a_r)=\Sigma B_r f(\xi\beta_r).$$

We have by integration

$$\Sigma A_r[\text{lin}\,p\xi-(\nu\xi a_r)]=-\Sigma B_r F(\xi\beta_r),\ \text{where}\ \frac{dF(y)}{dy}=f(y),$$

$$\therefore\ \text{lin}\,p\xi\,\Sigma A_r=\Sigma A_r(\nu\xi a_r)-\Sigma B_r F(\xi\beta_r),$$

which gives the value of $\text{lin}\,p\xi$.

4. Rectify the parabola $(\xi s)\sin(\xi\delta)=a$.

We have from this

$$\int(\xi s)\,d\xi=a\int\text{cosec}\,(\xi\delta)\,d\xi,$$

$$\therefore\ \text{lin}\,p\xi-(\nu\xi s)=-a\log\tan\tfrac{1}{2}(\xi\delta),$$

$$\therefore\ \text{lin}\,p\xi=(\xi s)\cot(\xi\delta)-a\log\tan\tfrac{1}{2}(\xi\delta).$$

5. Find the conic of closest contact of a curve.

The equation of a conic touching the tangent τx at x of a curve and also having the same curvature is

$$\rho x(y\nu x)^2+2h(y\nu x)(y\tau x)+b(y\tau x)^2=2(y\tau x).$$

Differentiating three times and putting $y=x$, we have by means of the formulae on p. 70

$$2h\{-3\rho x\}=2\{-\rho' x\},\ \therefore\ h=\frac{\rho' x}{3\rho x}.$$

Hence the family of conics having four point contact is

$$\rho x(y\nu x)^2+\frac{2\rho' x}{3\rho x}(y\nu x)(y\tau x)+b(y\tau x)^2=2(y\tau x).$$

Again, differentiating four times and putting $y=x$,

$$\rho x\{-8\rho x^2\}+2h\{-4\rho' x\}+b\{6\rho x^2\}=2(\rho x^3-\rho'' x).$$

Hence
$$b=\frac{5}{3}\rho x+\frac{4}{9}\frac{\rho' x^2}{\rho x^3}-\frac{1}{3}\frac{\rho'' x}{\rho x^2}.$$

Hence the conic of closest contact is

$$\rho x(y\nu x)^2+\frac{2\rho' x}{3\rho x}(y\nu x)(y\tau x)+\left\{\frac{5}{3}\rho x+\frac{4}{9}\frac{\rho' x^2}{\rho x^3}-\frac{1}{3}\frac{\rho'' x}{(\rho x)^2}\right\}(y\tau x)^2=2(y\tau x).$$

6. Similarly, in tangentials, shew that the conic of closest contact at ξ is

$$-9(\rho\xi)^4\sin^2(\eta\xi)+6\rho\xi^2(\eta p\xi)\{3\rho\xi\cos(\eta\xi)+\rho'\xi\sin(\eta\xi)\}$$
$$+(9\rho\xi^2-3\rho\xi\rho''\xi+4\rho'\xi^2)(\eta p\xi)^2=0.$$

7. From the equation $|(xs)|+|(xs')|=$ constant of a point of an ellipse, deduce directly that $(\xi s)(\xi s')=$constant of a line of an ellipse.

From $|(xs)|+|(xs')|=$constant we have

$$\frac{(\nu xs)}{|(xs)|}+\frac{(\nu xs')}{|(vs')|}=0,$$

$$\therefore\ \frac{(xs)^2}{(\nu xs)^2}=\frac{(xs')^2}{(\nu xs')^2},$$

from which
$$\frac{(\nu xs)}{(\tau xs)}+\frac{(\nu xs')}{(\tau xs')}=0,$$

i.e.
$$\frac{(\nu\xi s)}{(\xi s)}+\frac{(\nu\xi s')}{(\xi s')}=0,\ \text{putting } \tau x=\xi,$$

$\therefore$ integrating
$$\log(\xi s)+\log(\xi s')=\text{const}.$$

8. Shew that the join of the intersection of the normals at the extremities of a focal chord of an ellipse with the middle point of the chord is parallel to the major axis.

Let the ellipse be $|(xs)|=e(x\delta)$, s the focus, δ the directrix. Instead of differentiating we may obtain the equation of the tangent at y in the form of an equation.

For we have from Exs. 3, 4, p. 20,

$$\left.\begin{array}{l}|(\overline{xy}\,\overline{\zeta}s)|=|(ys)|-k|(xy)|\cos(\overline{xy}\,\overline{ys})\\ (\overline{xy}\zeta\delta)=(y\delta)+k\{(y\delta)-(x\delta)\}\end{array}\right\}\ k \text{ small.}$$

If the point $\overline{xy\zeta}$ is a point on the curve, then $\overline{xy}$ will be the tangent at x if $\dfrac{(y\zeta)}{(x\zeta)}=k$ is ultimately zero.

Employing this method

$$|(ys)|-k|(xy)|\cos(\overline{xy}\,\overline{ys})=e(y\delta)+ek\{(y\delta)-(x\delta)\}.$$

Hence the equation of the tangent is

$$(xy_{\overline{ys}\frac{\pi}{2}})-e(x\delta)+e(y\delta)=0.$$

This equation may also be obtained by differentiating $|(ys)|=e(y\delta)$ and putting $\tau y=\overline{xy}$.

If z be a point on the normal $\bar{z}\bar{y}$ is perpendicular to this.

Hence the equation of the normal is

$$\cos(z\bar{y}\bar{y}s_{\frac{\pi}{2}}) - e\cos(\bar{z}y\delta) = 0,$$

$$\therefore\ (zy\overline{ys}) + e\,(zy\delta_{\frac{\pi}{2}}) = 0,$$

i.e. $\quad (z\bar{y}\bar{s}) - e\,(za) + e\,(ya) = 0$ where a the major axis $= s_{\delta - \frac{\pi}{2}}$.

Similarly if y' be the other extremity of the curve and z now the intersection of the two normals

$$(z\overline{y'}s) - e\,(za) + e\,(y'a) = 0,$$

$\therefore$ adding $\quad (ya) + (y'a) = 2\,(za).$

9. *MacCullagh's Theorem.* If a chord pp' of a conic pass through a fixed point o, then

$$\tan\tfrac{1}{2}\,(\overline{ps}\overline{so})\tan\tfrac{1}{2}\,(\overline{p's}\overline{so}) = \text{constant}.$$

Let o_ω be the chord $\overline{pp'}$, and s_λ be $\overline{sp}$.

Then $\overline{o_\omega s_\lambda} = p$, so that

$$|(o_\omega s_\lambda s)| = e\,(o_\omega s_\lambda \delta).$$

Reducing $\quad (o_\omega s) = e\,\{(os_\lambda)\sin(\omega\delta) - (o\delta)\sin(\omega\lambda)\},$

supposing, as we may, $(\overline{pp'}s)$ to be positive.

$$\therefore\ \frac{1}{e}\,(so\omega) = (os\lambda)\sin(\omega\delta) + (o\delta)\,\{\sin(\bar{o}\bar{s}\lambda)\cos(s\bar{o}\omega) - \sin(\bar{s}\bar{o}\omega)\cos(\bar{o}s\lambda)\}.$$

Put $|(so)| = k\,(o\delta),$

$$\therefore\ \frac{k}{e} = y\sin(os\lambda) - \cos(os\lambda),$$

where y depends only on ω.

Writing $\quad x = \tan\tfrac{1}{2}\,(\bar{p}s\bar{s}o),\ \sin(\overline{os}\lambda) = \dfrac{2x}{1-x^2},\ \cos(\overline{os}\lambda) = \dfrac{1+x^2}{1-x^2}.$

Hence $\quad x^2(e-k) - 2xy + (e+k) = 0,$

an equation defining λ in terms of ω.

$$\therefore\ x_1x_2 = \frac{e+k}{e-k}.$$

The theorem is true when o traces a conic with the same focus and directrix as those of the given conic.

10. o and o' are two fixed points, x any point on the curve

$$\frac{1}{|(xo)|} - \frac{1}{|(xo')|} = \frac{1}{c}.$$

Prove that the distance between x and the consecutive curve obtained by changing c to $c + \delta c$ is ultimately

$$\frac{\delta c}{\sqrt{1 + \dfrac{3c^2}{rr'} + \dfrac{a^2c^4}{r^3r'^3}}},$$

where $\quad r = |(xo)|,\ r' = |(xo')|,\ a = |(oo')|.$ [Smith's Prize.]

Differentiating the equation

$$\frac{1}{|(xo)|}-\frac{1}{|(xo')|}=\frac{1}{c}$$

along the normal we have

$$\frac{1}{(xo)^2}\cos(\nu x\overline{xo})\,d\rho-\frac{1}{(xo')^2}\cos(\nu x\overline{xo'})\,d\rho=\frac{dc}{c^2},$$

where $d\rho$ is an element along the normal.

And we have also differentiating along the tangent

$$\frac{1}{(xo)^2}\cos(\tau x\overline{xo})-\frac{1}{|(x'o)|}\cos(\tau x\overline{xo'})=0.$$

These two equations give the required value for $d\rho$.

To eliminate τx, νx put $(\tau x\overline{xo})=\theta$, $(\tau x\overline{xo'})=\theta'$, and we have the following equations

$$\left.\begin{aligned}&\frac{\sin\theta}{r^2}-\frac{\sin\theta'}{r'^2}=\frac{1}{c^2}\frac{dc}{d\rho},\\&\frac{\cos\theta}{r^2}-\frac{\cos\theta'}{r'^2}=0,\\ \text{also}\quad&\theta-\theta'=\phi,\ \text{where}\ \phi=(\overline{xo'}\,\overline{xo})\end{aligned}\right\}.$$

Eliminating θ, θ' we have

$$\frac{\sqrt{r^4+r'^4-2r^2r'^2\cos\phi}}{r^2r'^2}=\frac{1}{c^2}\frac{dc}{d\rho},$$

$$\therefore\ d\rho=\frac{dc}{c^2}\frac{r^2r'^2}{\sqrt{r^4+r'^4-rr'(r^2+r'^2-a^2)}}$$

$$=\frac{dc}{\sqrt{1+\frac{3c^2}{rr'}+\frac{a^2c^4}{r^3r'^3}}},\ \text{by means of}\ r-r'=-\frac{rr'}{c}.$$

11. In a system of curves defined by an equation containing a variable parameter investigate at any point the normal distance between two curves. [Cayley.]

Take the general equation

$$f\{|(xa)|,\ |(xb)|,\ \ldots\ (xa),\ (x\beta)\ \ldots\}=c,$$

where c is a variable parameter.

Differentiating along the normal

$$d\rho\left\{\Sigma\frac{\partial f}{|(xa)|}\sin(\tau x\overline{xa})-\Sigma\frac{\partial f}{\partial(xa)}\cos(\tau xa)\right\}=dc,$$

and along the tangent

$$\Sigma\frac{\partial f}{\partial|(xa)|}\cos(\tau x\overline{xa})+\Sigma\frac{\partial f}{\partial(xa)}\sin(\tau xa)=0,$$

which two equations enable us to eliminate τx.

12. From the theorem that the circumcircle of a triangle circumscribing a parabola passes through the focus shew by differentiation that if an isosceles triangle circumscribe a parabola, the join of the vertex with the point of contact of the base is incident in the focus.

Let α, β, γ be the sides of any triangle circumscribing a parabola of which the focus is s. Then we have, since the circumcircle passes through s,

$$(s\beta)(s\gamma)\sin(\beta\gamma)+(s\gamma)(s\alpha)\sin(\gamma\alpha)+(s\alpha)(s\beta)\sin(\alpha\beta)=0.$$

Differentiate with regard to one of the lines, say γ, keeping the other elements fixed.

We have

$$(s\beta)\{(s\nu\gamma)\sin(\beta\gamma)+(s\gamma)\cos(\beta\gamma)\}+(s\alpha)\{(s\nu\gamma)\sin(\gamma\alpha)-(s\gamma)\cos(\gamma\alpha)\}=0\ldots(\text{A}).$$

The condition that s, the vertex $\overline{\alpha\beta}$, and the point of contact $p\gamma$ of γ should be collinear is

$$(\overline{\alpha\beta}\,s\;\overline{\gamma\nu\gamma})=0,$$

$$\therefore\ (\alpha\beta\gamma)(s\nu\gamma)-(\alpha\beta\nu\gamma)(s\gamma)=0,$$

$$\therefore\ (s\nu\gamma)\{(s\alpha)\sin(\beta\gamma)+(s\beta)\sin(\gamma\alpha)+(s\gamma)\sin(\alpha\beta)\}$$
$$-(s\gamma)\{(s\alpha)\sin(\beta\nu\gamma)+(s\beta)\sin(\nu\gamma\alpha)+(s\nu\gamma)\sin(\alpha\beta)\}=0.$$

Hence the condition is

$$\{(s\alpha)\sin(\beta\gamma)+(s\beta)\sin(\gamma\alpha)\}(s\nu\gamma)-(s\gamma)\{(s\alpha)\cos(\beta\gamma)-(s\beta)\cos(\gamma\alpha)\}=0\ldots(\text{B}).$$

(A) and (B) agree when $(\beta\gamma)=(\gamma\alpha)$.

13. [Bertrand.] If through each point of a curve a line of given length be drawn, making a constant angle with the normal of the curve, the normal to the locus of the extremity of this line passes through the corresponding centre of curvature of the proposed curve.

Consider the point $x_{\omega,c}$, where x is a point of the curve, c a constant and ω makes a constant angle with τx. We need the value of $(\nu x_{\omega,c}a)$.

From Ex. 1, § 55,

$$(\nu x_{\omega,c}a)=\{(\nu xa)+c\cos(\tau x\omega)\}\,dx+c\,(xa\omega)\,d\omega,$$

when c is constant.

$$\therefore\ (\nu x_{\omega,c}\,x_{\nu x,\frac{1}{\rho x}})=c\,\{\cos(\tau x\omega)\,dx+d\tau x\,(x\,x_{\nu x,\frac{1}{\rho x}}\,\omega)\},$$

since $(\omega\tau x)=$ constant,

$$=0.$$

14. If ξ_1, ξ_2, ... ξ_n be a set of parallel lines fixed in regard to the tangent and normal at a variable point x of a curve, shew that ξ_1, ξ_2, ... ξ_n envelope a set of parallel curves.

15. To find the polars of a point in regard to an algebraic curve.

Let $$P\{(xa)^2,\ (xb)^2\ldots\ (x\alpha),\ (x\beta)\ldots\}=0$$

be the curve, where P is a polynomial.

Let y be the point.

Let $\overline{yz}$ meet the curve in the point $\overline{yz}\xi$.

We have $$P\{(\overline{yz\xi}a)^2\ldots(\overline{yz\xi}\alpha)\ldots\}=0,$$

$$\therefore\ P\left\{\frac{(za)^2-k\{(ya)^2+(za)^2-(yz)^2\}+k^2(ya)^2}{(1-k)^2}\ldots\frac{(y\alpha)-k(z\alpha)}{1-k}\ldots\right\}=0,$$

where $$\frac{(z\xi)}{(y\xi)}=k.$$

Hence by putting the successive coefficients of $k=0$, we get equations in y, z. Looking upon z as a variable, these are the equations of the successive polars of y.

16. Find the poles of a line in regard to an algebraic curve.

Notation. We shall use the notation $a\,.\,b_{k_1/k_2}$ to denote the point $\overline{\overline{a}\overline{b}\gamma}$, where $\dfrac{(a\gamma)}{(b\gamma)}=\dfrac{k_1}{k_2}$.

17. Shew that $$(a\,.\,b_{k_1/k_2}\lambda)=\frac{(a\lambda)k_2-(b\lambda)k_1}{k_2-k_1}.$$

18. Shew that

$$\left(a\,.\,a_{1-k_1}\,.\,a_{2-\frac{k_2}{1+k_1}}\cdots a_{n-\frac{k_n}{1+\Sigma k_r}}\lambda\right)=\frac{(a\lambda)+\sum\limits_1^n(a_r\lambda)}{1+\sum\limits_1^n k_r}.$$

Put the L.H.S. $=P_n$.

Then $$P_n=\frac{P_{n-1}\left(1+\sum\limits_1^{n-1}k_r\right)+k_n(a_n\lambda)}{1+\sum\limits_1^n k_r},$$

$$\therefore\ P_n\left(1+\sum_1^n k_r\right)-P_{n-1}\left(1+\sum_1^{n-1}k_r\right)=k_n(a_n\lambda).$$

Similarly $$P_{n-1}\left(1+\sum_1^{n-1}k_r\right)-P_{n-2}\left(1+\sum_1^{n-2}k_r\right)=k_{n-1}(a_{n-1}\lambda),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$P_2\left(1+\sum_1^2 k_r\right)-P_1(1+k_1)=k_2(a_2\lambda),$$

$\therefore$ adding $$P_n\left(1+\sum_1^n k_r\right)-P_1(1+k_1)=\sum_2^n k_r(a_r\lambda),$$

which gives P_n.

Notation. We shall use the notation $a\,.\,\beta_{k_1/k_2}$ to denote the line $\overline{\overline{a}\overline{\beta}c}$, where $\dfrac{(ac)}{(\beta c)}=\dfrac{k_1}{k_2}$. It is evident that this does not completely define the line, as it does not specify any sense.

19. Shew that $$(a\,.\,\beta_{k_1/k_2}c)=\frac{k_2(ac)-k_1(\beta c)}{\sqrt{k_1^2+k_2^2-2k_1k_2\cos(a\beta)}},$$
where the sign of the square root is arbitrary.

20. If d be the isotomic conjugate of c in regard to a, b; a, b, c being coincident: find $(d\lambda)$.

If δ be the isogonal conjugate of γ in regard to a, β; a, β, γ being coincident: find (δl).

Point Reciprocation. Let o be a fixed point.

Let x be any point. The reciprocal of x in regard to o is defined as ξ, where

$$\xi = o_{\overline{ox},\ \frac{K}{|(ox)|};\ \overline{ox}_{\frac{\pi}{2}}} \quad \text{where } K \text{ is a constant;}$$

from which it is easy to shew that $x = o_{\xi_{-\frac{\pi}{2}},\ \frac{K}{(o\xi)}}$.

21. Shew that, if ξ, η be the reciprocals of x, y in regard to o,

$$\sin(\xi\eta) = |(xy)| \cdot \frac{(o\bar{x}\bar{y})}{|(ox)(oy)|},$$

$$\sin(\xi\eta) = \sin(\overline{ox}\,\overline{oy})$$

$$= \frac{(oxy)}{|(yo)(ox)|} = |(xy)| \cdot \frac{(o\bar{x}\bar{y})}{|(xo)(yo)|}.$$

22. Shew that

$$|(xy)| = K\frac{(\xi\eta o)}{(o\xi)(o\eta)}.$$

We have

$$|(xy)| = |(o_{\xi_{-\frac{\pi}{2}},\ \frac{K}{(o\xi)}}\, y)|$$

$$= \left|\left\{(oy)^2 - 2\frac{K}{(o\xi)}(oy\xi) + \frac{K^2}{(o\xi)^2}\right\}^{\frac{1}{2}}\right|$$

$$= \left|\left\{(o_{\eta_{-\frac{\pi}{2}},\ \frac{K}{(o\eta)}}\, o)^2 - 2\frac{K}{(o\xi)}\{(o\xi) - (o_{\eta_{-\frac{\pi}{2}},\ \frac{K}{(o\eta)}}\, \xi)\} + \frac{K^2}{(o\xi)^2}\right\}^{\frac{1}{2}}\right|$$

$$= \left|\left\{\frac{K^2}{(o\eta)^2} - \frac{2K}{(o\xi)}\left\{(o\xi) - (o\xi) + \sin(\eta_{-\frac{\pi}{2}}\xi)\frac{K}{(o\eta)}\right\} + \frac{K^2}{(o\xi)^2}\right\}^{\frac{1}{2}}\right|$$

$$= \frac{|K|}{(o\xi)(o\eta)}\,|\{(o\xi)^2 + (o\eta)^2 - 2(o\xi)(o\eta)\cos(\xi\eta)\}^{\frac{1}{2}}|.$$

23. Shew that $\displaystyle (x\eta) = K\frac{(y\xi)}{|(oy)|(o\xi)}$.

$$(x\eta) = (o_{\xi_{-\frac{\pi}{2}},\ \frac{K}{(o\xi)}}\ o_{\overline{oy},\ \frac{K}{|(oy)|},\ \overline{oy}_{\frac{\pi}{2}}})$$

$$= (o\ o_{\overline{oy},\ \frac{K}{|(oy)|};\ \overline{oy}_{\frac{\pi}{2}}}) - \frac{K}{(o\xi)}\sin(\xi_{-\frac{\pi}{2}}\overline{oy}_{\frac{\pi}{2}})$$

$$= \frac{K}{|(oy)|} - \frac{K}{(o\xi)}\sin(\overline{oy}\,\xi)$$

$$= \frac{K}{|(oy)|(o\xi)}[(o\xi) - (oy\xi)]$$

$$= \frac{K(y\xi)}{|(oy)|(o\xi)}.$$

For point reciprocation, we shall denote by R_o the reciprocal of any element in regard to o.

The preceding formulae become

$$\sin(R_o x\, R_o y) = \frac{(oxy)}{|(ox)\,(oy)|},$$

$$|(R_o\xi\, R_o\eta)| = K \cdot \frac{(\xi\eta o)}{(o\xi)\,(o\eta)},$$

$$(R_o\xi\, R_o y) = K \cdot \frac{(\xi y)}{(o\xi)\,|(oy)|}.$$

24. Similarly shew that

$$(R_o\xi\, R_o\eta\, R_o\zeta) = K \cdot \frac{(\xi\eta\zeta)}{(o\xi)\,(o\eta)\,(o\zeta)},$$

$$(R_o x\, R_o y\, R_o z) = K \cdot \frac{(xyz)}{|(ox)\,(oy)\,(oz)|},$$

$$(R_o\xi\, R_o\eta\, R_o z) = K \cdot \frac{(\xi\eta z o)}{|(oz)|\,(o\eta)\,(o\xi)},$$

$$(R_o x\, R_o y\, R_o\zeta) = K \cdot \frac{(xy\zeta o)}{(o\zeta)\,|(ox)\,(oy)|}.$$

25. Shew generally that

$$(x_1 x_2 \xi_1 x_3 \xi_2 \ldots x_n x_{n+1}) = K \cdot \frac{(R_o x_1\, R_o x_2\, R_o\xi_1\, R_o x_3\, R_o\xi_2 \ldots R_o x_n\, R_o x_{n+1})}{(oR_o x_1)\,(oR_o x_2)\,|(oR_o\xi_1)| \ldots (oR_o x_{n+1})},$$

$$(x_1 x_2 \xi_1 x_3 \xi_2 \ldots x_n \xi_{n-1}) = K \cdot \frac{(R_o x_1\, R_o x_2\, R_o\xi_1\, R_o x_3 \ldots R_o x_n\, R_o\xi_{n-1}\, o)}{(oR_o x_1)\,(oR_o x_2)\,|(oR_o\xi_1)| \ldots (oR_o x_{n+1})},$$

$$(\xi_1 \xi_2 x_1 \xi_3 x_2 \ldots \xi_n \xi_{n+1}) = K \cdot \frac{(R_o\xi_1\, R_o\xi_2\, R_o x_1 \ldots R_o\xi_n\, R_o\xi_{n+1})}{|(oR_o\xi_1)\,(oR_o\xi_2)|\,(oR_o x_1) \ldots |(oR_o\xi_{n+1})|},$$

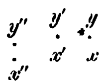
$$(\xi_1 \xi_2 x_1 \xi_3 x_2 \ldots \xi_n x_{n-1}) = K \cdot \frac{(R_o\xi_1\, R_o\xi_2\, R_o x_1\, R_o\xi_3\, R_o x_2 \ldots R_o\xi_n\, R_o x_{n-1}\, o)}{|(oR_o\xi_1)\,(oR_o\xi_2)|\,(oR_o x_1) \ldots |(oR_o x_{n-1})|}.$$

26. *Roulettes.*

One curve rolls on another fixed curve, to find the displacement of the point of contact and the tangent at the point of contact on the rolling curve. Suppose the curve to roll counter clockwise. Let the senses of description of the curves be counter clockwise.

Let x, x', x'' be three contiguous points on one curve, y, y', y'' three points on the moving curve which take up positions x, x', x'' in its rolling.

Let $$x = y, \quad x' = y',$$



In the rolling y' remains at x', but $\overline{y'y''}$ becomes $\overline{x'x''}$.

I.e. if y be the point of contact on the rolling curve, then

$$dy = 0, \quad d\tau y = -d\tau_1 y + d\tau x, \quad d\tau x > d\tau_1 y,$$

where $d\tau_1 y$ is the displacement of τy when the curve is fixed.

Knowing these two displacements, we may find the displacement of any element derived vectorially or equationally from them, by the method of the text.

The case of a point on a rolling curve does not come under the classes of derived points considered. The above investigation from first principles is therefore necessary.

27. Find the displacement of a point fixed in regard to the rolling curve.

Let z be a point fixed in regard to the rolling curve and y the point of contact on the rolling curve.

Let
$$z = y_{\omega, R},$$
$$\therefore\ dz = Rd\omega = Rd\tau y = R\,(d\tau x - d\tau_1 y),$$
and
$$\tau z = \overline{yz}_{\frac{\pi}{2}}.$$

28. Find the displacement of a carried line.

Let ζ be the line.

Then
$$d\zeta = d\tau y = d\tau x - d\tau_1 y,$$
and
$$p\zeta = y_{\zeta_{\frac{\pi}{2}}, -(y\zeta)}.$$

29. Find the radius of curvature of a carried point.

We may now no longer concern ourselves with the fixed curve.

Let z be the point.

Then
$$d\tau z = dyz \text{ by Ex. 27}$$
$$= \frac{\sin(\tau y\,\overline{yz})}{|(yz)|}\,dy - \frac{\sin(\tau z\,\overline{yz})}{|(yz)|}\,dz.$$

Here the displacement of y has a different significance from what it has in Ex. 26. In Ex. 26 the considerations of its displacement were due to the rolling. The displacement now is due to the point taking up, as we suppose, successive positions on the curve.

If
$$z = y_{\omega, R},$$
$$d\tau y_{\omega, R} = \frac{\sin(\tau y\omega)}{R}\,dy + \frac{1}{R}\,R\,(d\tau x - d\tau_1 y),$$
$$\therefore\ \rho y_{\omega, R} = \frac{\sin(\tau y\omega)\,dy}{Rdy_{\omega, R}} + \frac{d\tau x - d\tau_1 y}{dy_{\omega, R}}$$
$$= \frac{\sin(\tau y\omega)\,dy}{R^2\,(d\tau x - d\tau_1 y)} + \frac{1}{R}$$
$$= \frac{\sin(\tau y\omega)}{R^2\,(\rho x - \rho y)} + \frac{1}{R}.$$

By similarly differentiating
$$\rho z = \frac{\sin(\tau y\,\overline{yz})}{(yz)^2\,(\rho x - \rho y)} + \frac{1}{|(yz)|},$$
find $\dfrac{d\rho z}{dz}$: and so on.

30. **Find the radius of curvature of a carried line.**

Let ζ be the line.

Then
$$\nu\zeta = y_{\zeta_{\frac{\pi}{2}}},$$
$$(\nu^2\zeta a) = \frac{d(\nu\zeta a)}{d\nu\zeta} = \frac{d(y_{\zeta_{\frac{\pi}{2}}}a)}{d\zeta} - = \frac{d(ay\zeta_{\frac{\pi}{2}})}{d\zeta}$$
$$= \sin(\tau y\zeta_{\frac{\pi}{2}})\frac{dy}{d\zeta} - (ay\zeta)$$
$$= \cos(\tau y\zeta)\frac{dy}{d\tau x - d\tau_1 y} - (ay\zeta),$$
$$\therefore\ \rho\zeta = (\nu^2\zeta p\zeta) = \frac{\cos(\tau y\zeta)}{\rho x - \rho y} + (y\zeta).$$

Consider the case of a curve rolling on another curve which is rolling on another curve.

31. Shew that the pedal triangles of a triangle of points inverse in regard to the circumcircle are similar.

Let the points be $s_{\omega,\rho}$, $s_{\omega,\frac{R^2}{\rho}}$.

Now if x, y, z be the summits of the pedal triangle of $s_{\omega,\rho}$,
$$(yz)^2 = (oa)^2\sin^2(\beta\gamma)$$
$$= (s_{\omega,\rho}a)^2\sin^2(\beta\gamma)$$
$$= \{R^2 + \rho^2 - 2R\rho\cos(\overline{sa}\,\omega)\}\sin^2(\beta\gamma).$$

If x', y', z' are the summits for $s_{\omega,\frac{R^2}{\rho}}$,
$$(y'z')^2 = \left\{\frac{R^4}{\rho^2} + R^2 - 2\frac{R^2}{\rho}\cos(\overline{sa}\,\omega)\right\}\sin^2(\beta\gamma)$$
$$= \frac{R^2}{\rho^2}(yz)^2.$$

32. If we represent by $I_o x$ the inverse point of x in regard to a circle centre o, shew that
$$|(xy)| = \frac{|(I_o x\, I_o y)|\,R^2}{|(oI_o x)(oI_o y)|}.$$

$I_o x = o_{\overline{ox},\frac{R^2}{|(ox)|}}$, where R is the radius of the circle.
$$\therefore\ (I_o x\, I_o y)^2 = (o_{\overline{ox},\frac{R^2}{|(ox)|}}\ o_{\overline{oy},\frac{R^2}{|(oy)|}})^2$$
$$= \frac{R^4}{(ox)^2} + \frac{R^4}{(oy)^2} - 2\frac{R^4}{|(ox)(oy)|}\cos(\overline{ox}\,\overline{oy})$$
$$= \frac{R^4(xy)^2}{(ox)^2(oy)^2}.$$

33. Shew that
$$(xyz) = \frac{(I_o x\, I_o y\, I_o z)(R_1^2 - \rho^2)}{(oI_o x)^2(oI_o y)^2(oI_o z)^2},$$
where R_1 is the circumradius of $I_o x$, $I_o y$, $I_o z$ and ρ the distance of o from the circumcentre of $I_o x$, $I_o y$, $I_o z$.

We have
$$(I_0x\, I_0y\, I_0z) = \left(o_{\overline{ox}},\ \frac{R^2}{|(ox)|}\ \ o_{\overline{oy}},\ \frac{R^2}{|(oy)|}\ \ o_{\overline{oz}},\ \frac{R^2}{|(oz)|}\right)$$
$$= \Sigma \frac{R^4}{|(oy)\,(oz)|} \sin(\overline{oy}\,\overline{oz})$$
$$= \frac{R^4}{(ox)^2\,(oy)^2\,(oz)^2} \Sigma\,(ox)^2\,(oyz)$$
$$= \frac{R^4}{(ox)^2\,(oy)^2\,(oz)^2}(xyz)(R_1{}^2 - \rho^2),$$
by theorem on p. 29 which proves the result.

Anharmonic or Cross-ratio.

Let a, b, c, d be four points incident in a line λ. Then the ratio
$$\frac{(ac)}{(ad)} \Big/ \frac{(bc)}{(bd)}$$
is called the anharmonic ratio or cross-ratio of the range of points, and is represented by $\{ab, cd\}$. If α, β, γ, δ be four lines incident in a point l, then the ratio $\dfrac{\sin(\alpha\gamma)}{\sin(\alpha\delta)} \Big/ \dfrac{\sin(\beta\gamma)}{\sin(\beta\delta)}$ is called the anharmonic ratio or cross-ratio of the pencil of lines and is represented by $\{\alpha\beta, \gamma\delta\}$.

In projective geometry, of the trigonometric functions the sine function only occurs; hence, for brevity, we shall represent $\sin(\alpha\beta)$ by $(\alpha\beta)$.

Thus the cross-ratio of four lines incident in a point is
$$\frac{(\alpha\gamma)}{(\alpha\delta)} \Big/ \frac{(\beta\gamma)}{(\beta\delta)}.$$

The cross-ratio of a pair of points a, b and a pair of lines γ, δ we shall define as
$$\frac{(a\gamma)}{(a\delta)} \Big/ \frac{(b\gamma)}{(b\delta)}\,^*,$$
and this is written $\{ab, \gamma\delta\}$.

34. Reduce $\{\overline{aa'}\,\overline{bb'},\ \overline{\gamma\gamma'}\,\overline{\delta\delta'}\}$.
$$\{\overline{aa'}\,\overline{bb'},\ \overline{\gamma\gamma'}\,\overline{\delta\delta'}\} = \frac{(\overline{aa'}\,\overline{\gamma\gamma'})}{(\overline{aa'}\,\overline{\delta\delta'})} \Big/ \frac{(\overline{bb'}\,\overline{\gamma\gamma'})}{(\overline{bb'}\,\overline{\delta\delta'})}$$
$$= \frac{(a\gamma)(a'\gamma') - (a\gamma')(a'\gamma)}{(a\delta)(a'\delta') - (a\delta')(a'\delta)} \Big/ \frac{(b\gamma)(b'\gamma') - (b\gamma')(b'\gamma)}{(b\delta)(b'\delta') - (b\delta')(b'\delta)}.$$

Particular cases.

When $(a\gamma') = 0$, $(a'\delta) = 0$; $(b\gamma') = 0$, $(b'\delta) = 0$.

Then $\{\overline{aa'}\,\overline{bb'},\ \overline{\gamma\gamma'}\,\overline{\delta\delta'}\} = \{ab, \gamma\delta\} \cdot \{a'b', \gamma'\delta'\}$.

Similarly if $(a\gamma') = 0$, $(a'\delta) = 0$; $(b\gamma) = 0$, $(b'\delta') = 0$,

then $\{\overline{aa'}\,\overline{bb'},\ \overline{\gamma\gamma'}\,\overline{\delta\delta'}\} = \{ab', \gamma\delta\} \cdot \{a'b, \gamma'\delta'\}$.

* This ratio and its usage are new, as far as I know

35. If $\left.\begin{matrix} a & b & c \\ \alpha & \beta & \gamma \end{matrix}\right\}$, $\left.\begin{matrix} a' & b' & c' \\ \alpha' & \beta' & \gamma' \end{matrix}\right\}$ be two triangles; and $\overline{aa'}$, $\overline{bb'}$, $\overline{cc'}$ be represented by λ, μ, ν and $\overline{\alpha\alpha'}$, $\overline{\beta\beta'}$, $\overline{\gamma\gamma'}$ by l, m, n; shew that

$$\{\mu\nu, \alpha\alpha'\} . \{mn, aa'\} = 1.$$

$$\{\mu\nu, \alpha\alpha'\} = \{\overline{bb'}\,\overline{cc'}, \overline{\beta\gamma}, \overline{\beta'\gamma'}\}$$

$$= \frac{-(b\gamma)(b'\beta)}{(c\beta)(c'\gamma)} \Big/ \frac{(b\beta')(b'\gamma')}{-(c\gamma')(c'\beta')}$$

$$= \{b'c, \beta\gamma'\}/\{bc', \beta'\gamma\}.$$

Similarly $\qquad \{mn, aa'\} = \{\beta'\gamma, bc'\}/\{\beta\gamma', b'c\}.$

36. Shew that

$$(\overline{\overline{ab}\gamma}\;\overline{\overline{a'b'}\gamma'}, \delta\delta') = \frac{(a\gamma)(b\delta) - (a\delta)(b\gamma)}{(a\gamma)(b\delta') - (a\delta')(b\gamma)} \Big/ \frac{(a'\gamma')(b'\delta) - (a'\delta)(b'\gamma')}{(a'\gamma')(b'\delta') - (a'\delta')(b'\gamma')}.$$

37. In Ex. 35 shew that

$$\{mn, aa'\} . l = \{\mu\nu, \alpha\alpha'\} . \lambda,$$

where $\{ab, cd\} . o$ denotes $\{\overline{oa}\,\overline{ob}, \overline{oc}\,\overline{od}\}$.

$$\{mn, aa'\} . l = \{mn, \overline{la}\,\overline{la'}\} = \{\overline{\beta\beta'}\,\overline{\gamma\gamma'}, \overline{la}\;\overline{l'a}\}$$

$$= \frac{(\beta l)(\beta' a)}{(\beta a')(\beta' l)} \cdot \frac{(\gamma a')(\gamma' l)}{(\gamma l)(\gamma' a)}$$

$$= \{\beta\gamma, la'\}/\{\beta'\gamma', la\}.$$

Now

$$(\beta l) = (\overline{ca}\,\overline{aa'}) = -\frac{(ca')(aa)}{|(ca)|\,(aa')},$$

$$(\beta' l) = (\overline{c'a'}\,\overline{aa'}) = \frac{(c'a)(a'a')}{|(c'a')|\,(aa')},$$

$$(\gamma l) = (\overline{ab}\,\overline{aa'}) = \frac{(aa)(ba')}{|(ab)|\,(aa')},$$

$$(\gamma' l) = (\overline{a'b'}\,\overline{aa'}) = -\frac{(a'a')(b'a)}{|(a'b')|\,(aa')}.$$

Hence $\qquad \{mn, aa'\} . l = \frac{(ca')(\gamma a') . (b'a)(\beta' a)}{(c'a)(\gamma' a) . (ba')(\beta a')} \cdot \left|\frac{(c'a')\,(ab)}{(a'b')\,(ca)}\right|,$

which proves the result.

38. In the two triangles $\left.\begin{matrix} abc \\ \alpha\beta\gamma \end{matrix}\right\}$, $\left.\begin{matrix} a'b'c' \\ \alpha'\beta'\gamma' \end{matrix}\right\}$ investigate the measure

$$(\overline{\overline{bc'}\,\overline{b'c}}\;\overline{\overline{ca'}\,\overline{c'a}}\;\overline{\overline{ab'}\,\overline{a'b}}).$$

We have $\quad (\overline{\overline{bc'}\,\overline{b'c}}\;\overline{\overline{ca'}\,\overline{c'a}}\;\overline{\overline{ab'}\,\overline{a'b}})\,(\overline{bc'}\,\overline{b'c})\,(\overline{ca'}\,\overline{c'a})\,(\overline{ab'}\,\overline{a'b})$

$= (\overline{bc'}\,\overline{ca'}\,\overline{c'a})\,(\overline{b'c}\,\overline{ab'}\,\overline{a'b}) - (\overline{bc'}\,\overline{ab'}\,\overline{a'b})\,(\overline{b'c}\,\overline{ca'}\,\overline{c'a})$

$= \{(bc'a)(cc'a')(b'a'b)(cab')$

$\qquad - (bab')(c'a'b)(b'ca')(cc'a)\}/|(bc')(ca')(a'b)(b'c)(c'a)(a'b)|$

$= \{(c'\gamma)(c\beta')(b\gamma')(b'\beta) - (b'\gamma)(b\beta')(c\gamma')(c'\beta)\}\ \dfrac{|(ca)(c'a)(ab)(a'b')|}{|(bc')(ca')(ab')(b'c)(c'a)(a'b)|}.$

Corollary 1. If $(\overline{\overline{bc'}\,\overline{b'c}}\ \overline{\overline{ca'}\,\overline{c'a}}\ \overline{\overline{ab'}\,\overline{a'b}})=0$ then

$$(c'\gamma)(c\beta')(b\gamma')(b'\beta)=(b'\gamma)(b\beta')(c\gamma')(c'\beta),$$

i.e.
$$\{bc,\ \beta'\gamma'\}=\{b'c',\ \beta\gamma\}.$$

Similarly
$$\{ca,\ \gamma'a'\}=\{c'a',\ \gamma a\},$$
$$\{ab,\ a'\beta'\}=\{a'b',\ a\beta\}.$$

Corollary 2. Or if $\{bc,\ \beta'\gamma'\}=\{b'c',\ \beta\gamma\}$ then

$$\{ca,\ \gamma'a'\}=\{c'a',\ \gamma a\},$$

and
$$\{ab,\ a'\beta'\}=\{a'b',\ a\beta\}.$$

Also
$$(\overline{\overline{bc'}\,\overline{b'c}}\ \overline{\overline{ca'}\,\overline{c'a}}\ \overline{\overline{ab'}\,\overline{a'b}})=0.$$

39. Investigate $(\overline{\overline{bc'}\,\overline{b'c}}\ \overline{\overline{ca}\,\overline{c'a'}}\ \overline{\overline{ab}\,\overline{a'b'}})$.

Expression
$$=(\overline{\overline{bc'}\,\overline{b'c}}\ \overline{\beta\beta'}\ \overline{\gamma\gamma'})$$

$$=\frac{1}{|(bc')(b'c)|}\frac{(b\beta)(c'\beta')-(b\beta')(c'\beta)}{(b\gamma)(c'\gamma')-(b\gamma')(c'\gamma)}\frac{(b'\gamma)(c\gamma')-(b'\gamma')(c\gamma)}{(b'\beta)(c\beta')-(b'\beta')(c\beta)}$$

$$=\frac{1}{|(bc')(b'c)|}\frac{(b\beta')(c'\beta)}{(b\gamma')(c'\gamma)}\cdot\frac{(b'\gamma)(c\gamma')}{(b'\beta)(c\beta')}$$

$$=\frac{1}{|(bc')(b'c)|}\frac{\{bc,\ \beta'\gamma'\}}{\{b'c',\ \beta\gamma\}}.$$

Corollary. From Exs. 38 and 39 if

$$(\overline{\overline{bc'}\,\overline{b'c}}\ \overline{\overline{ca'}\,\overline{c'a}}\ \overline{\overline{ab'}\,\overline{a'b}})=0,$$

then
$$(\overline{\overline{bc'}\,\overline{b'c}}\ \overline{\overline{ca}\,\overline{c'a'}}\ \overline{\overline{ab}\,\overline{a'b'}})=0.$$

40. If ab, cd be two pairs of points and x a variable point, such that

$$\{ab,\ cd\}\,.\,x \text{ is const.},$$

shew that the locus of x is a conic.

$$\{ab,\ cd\}\,.\,x=\frac{(xac)}{(xbc)}\Big/\frac{(xad)}{(xbd)}.$$

41. If ab, $\gamma\delta$ be a pair of points and a pair of lines: and $a\beta$, cd their reciprocals in regard to a conic, shew that

$$\{a\beta,\ cd\}=\{ab,\ \gamma\delta\}.$$

42. If $\left.\begin{matrix}abc\\ a\beta\gamma\end{matrix}\right\}$, $\left.\begin{matrix}a'b'c'\\ a'\beta'\gamma'\end{matrix}\right\}$ be two triangles, self-conjugate in regard to a conic, shew that the vertices lie on a conic, also that the sides touch a conic.

We have to shew that

$$a\,.\,\{bc,\ b'c'\}=a'\,.\,\{bc,\ b'c'\},$$

or
$$\{\beta\gamma,\ b'c'\}=\{bc,\ \beta'\gamma'\},$$

which follows since the triangles are self-conjugate: similarly the sides touch a conic.

43. Shew that if the vertices of two triangles touch a conic, then the triangles are self-conjugate for a conic.

Let Γ be a conic, for which $\left.\begin{matrix} abc \\ \alpha\beta\gamma \end{matrix}\right\}$ is self-conjugate and a', α' pole and polar. Then the polar of b' passes through a'; let it meet α' in c''.

Then $\left.\begin{matrix} abc \\ \alpha\beta\gamma \end{matrix}\right\}$, $\left.\begin{matrix} a'b'c'' \\ \alpha'\beta'\gamma' \end{matrix}\right\}$ are self-conjugate for Γ, $\therefore$ abc, $a'b'c''$ are on a conic.

Hence $c''=c'$.

44. If two triangles $\left.\begin{matrix} abc \\ \alpha\beta\gamma \end{matrix}\right\}$, $\left.\begin{matrix} a'b'c' \\ \alpha'\beta'\gamma' \end{matrix}\right\}$ are reciprocal one for the other, in regard to a conic, shew that the triangles are homological.

$$\{bc,\ \gamma'\alpha'\}=\{ca,\ \beta'\gamma'\},$$

$$\frac{(b\gamma')}{(b\alpha')}\Big/\frac{(c\gamma')}{(c\alpha')}=\frac{(c\beta')}{(c\gamma')}\Big/\frac{(a\beta')}{(a\gamma')},$$

$$\therefore\ (b\gamma')(c\alpha')(a\beta')=(c\beta')(a\gamma')(b\alpha').$$

45. Shew that $(xa\beta c\delta ex)=0$ is the equation of a conic · deduce Pascal's theorem.

By reducing $(xa\beta c\delta ex)=0$ we can prove the first part.

It is evident that a, e are points of the conic.

Next to find where β meets the curve.

Let $\beta=\overline{pq}$, and p be on the curve.

Then $$(pa\overline{pq}c\delta ep)=0.$$

$$\therefore\ (pc\delta ep)=0,\ \text{since}\ (a\beta)\neq 0$$

$$\therefore\ (pce)(\delta p)=0,$$

$$\therefore\ p=\overline{\beta\delta}\ \text{or}\ \overline{ce}\beta.$$

Hence a, e; $\overline{\beta\delta}$; $\overline{\overline{ce}\beta}$, $\overline{\overline{ac}\delta}$ are points on the conic

Let $$\overline{\beta\delta}=l,\ \overline{ce\beta}=m,\ \overline{\overline{ac}\delta}=n$$

Then $$\beta=\overline{lm},\ \delta=\overline{ln},\ c=\overline{\overline{em}\,\overline{an}}.$$

Hence if g be a point on the conic through a, e, l, m, n

$$(ga\overline{lm}\ \overline{\overline{em}\,\overline{an}}\ \overline{ln}eg)=0,$$

$$\therefore\ (\overline{ga}\,\overline{lm}\ \overline{em}\,\overline{an}\ \overline{ln}\,\overline{eg})=0,$$

which is Pascal's theorem*.

46. Shew that

$$(x\bar{a}\,a\,a_1\ \overline{xb}\,\beta\,k\,\gamma\,b_1\ \overline{xc})=0,$$

where $(\alpha\beta\gamma)=0$, $(aa_1\beta b_1k)=0$ denotes a general cubic curve, i.e. that it can be made to pass through nine arbitrary points†.

The curve obviously passes through a, b, c.

* See Whitehead's *Universal Algebra*, p. 232. † *Ibid.* pp. 234, 237.

Consider the point $\overline{\beta\gamma}=\overline{\gamma\alpha}=\overline{a\beta}$. It lies on the curve, for substituting this point for x we have

$$(\overline{a\beta}\,a\,a\,a_1\ \ \overline{\overline{a\beta}\,b\,\beta}\,k\,\gamma\,b_1\ \ \overline{a\beta}\,c)$$

$$=(\overline{a\beta}\,a_1\ \ \overline{a\gamma}\,k\,\gamma\,b_1\ \ \overline{a\beta}\,c)$$

$$=(\overline{a\beta}\,a_1\ \ \overline{a\beta}\,b_1\ \ \overline{a\beta}\,c)=0.$$

Again, consider the point $\overline{a_1c}\,a$.

$$(\overline{\overline{a_1c}\,a}\,a\,a\,a_1\ \ \overline{\overline{a_1c}\,a\,b\,\beta\,k\,\gamma\,b_1}\ \ \overline{\overline{a_1c}\,a}\,c)$$

$$=(\overline{a_1c}\ \ \overline{\overline{a_1c}\,a\,b\,\beta\,k\,\gamma\,b_1}\ \ \overline{a_1c})=0.$$

Again, consider the point $\overline{aa_1}\beta$.

$$(\overline{\overline{aa_1}\,\beta}\,a\,a\,a_1\ \ \overline{\overline{aa_1}\,\beta\,b\,\beta\,k\,\gamma\,b_1}\ \ \overline{\overline{aa_1}\,\beta}\,c)$$

$$=(\overline{aa_1}\ \ \overline{\overline{aa_1}\,\beta\,k\,\gamma\,b_1}\ \ \overline{\overline{aa_1}\,\beta}\,c)$$

$$=(\overline{aa_1}\,\beta\ \ \overline{\overline{aa_1}\,\beta\,k\,\gamma\,b_1})\times\sin(\overline{aa_1}\ \ \overline{\overline{aa_1}\,\beta}\,c)$$

$$=(f\ \overline{fk\gamma\,b_1})\times\{\ldots\},\ \text{where } f=\overline{aa_1}\beta$$

$$=0,\ \text{since } (f\,b_1\,k)=0.$$

Hence the six points a, b, c; d, e, f lie on the curve, where

$$d=\overline{a_1c}\,a,\quad e=\overline{a\beta}=\overline{\beta\gamma}=\overline{\gamma a},\quad f=\overline{aa_1}\beta.$$

Hence $$a=\overline{de},\quad \beta=\overline{ef}.$$

Also $$d=\overline{a_1c}\,a=\overline{a_1c}\,\overline{de},$$

$$\therefore\ (a_1cd)=0,$$

and $$f=\overline{a_1a}\,\beta=\overline{a_1a}\,\overline{ef},$$

$$\therefore\ (a_1af)=0,$$

$$\therefore\ a_1=\overline{\overline{af}\,\overline{cd}}.$$

Hence the cubic

$$(\overline{xa}\,\overline{de}\ \ \overline{\overline{af}\,\overline{cd}}\ \ \overline{\overline{xb}\,\overline{ef}\,k\,\gamma\,b_1}\ \ \overline{xc})=0$$

passes through a, b, c; d, e, f.

As regards k, γ, b_1, we have

$$(\gamma e)=0,\ (fb_1k)=0$$

Take three other points g, h, i.

Now let
$$g_1=\overline{\overline{\overline{ga}\,a\,a_1}\;\overline{gc}}=\overline{\overline{\overline{\overline{ga}\,de}\;\overline{af}}\;\overline{cd}\;\overline{gc}},$$
$$h_1=\overline{\overline{\overline{ha}\,a\,a_1}\;\overline{hc}}=\overline{\overline{\overline{\overline{ha}\,de}\;\overline{af}}\;\overline{cd}\;\overline{hc}},$$
$$i_1=\overline{\overline{\overline{ia}\,a\,a_1}\;\overline{ic}}=\overline{\overline{\overline{\overline{ia}\,de}\;\overline{af}}\;\overline{cd}\;\overline{ic}},$$
$$g_2=\overline{\overline{gb}\,\beta}=\overline{\overline{gb}\;\overline{ef}},$$
$$h_2=\overline{\overline{hb}\,\beta}=\overline{\overline{hb}\;\overline{ef}},$$
$$i_2=\overline{\overline{ib}\,\beta}=\overline{\overline{ib}\;\overline{ef}}.$$

Thus the six points g_1, h_1, i_1 ; g_2, h_2, i_2 can be obtained by linear construction from the nine points
$$a,\ b,\ c\ ;\ d,\ e,\ f;\ g,\ h,\ i.$$
We proceed to choose k, γ, b_1 so that the following equations hold:
$$(g_2k\gamma b_1g_1)=0,\ (h_2k\gamma b_1h_1)=0,\ (i_2k\gamma b_1i_1)=0,$$
which are the conditions that g, h, i should lie on the curve.

If possible, determine γ and k from
$$(i_1b_1\gamma ki_2)=0,$$
without conditioning b_1.

For this we must suppose $\overline{\overline{i_1b_1}\gamma}=i_1$, and $(i_1ki_2)=0$.

Hence
$$(\gamma i_1)=0.$$
Hence since $(\gamma e)=0$ as well
$$\gamma=\overline{i_1e}.$$
$\gamma=\overline{i_1e}$ and $(ki_1i_2)=0$ account for the first equation.

The remaining two equations can be written
$$(kg_2\gamma g_1b_1)=0,\ (kh_2\gamma h_1b_1)=0.$$
Hence k is such that
$$(\overline{\overline{kg_2}\gamma g_1}\;\overline{\overline{kh_2}\gamma h_1}\;\overline{kf})=0,\ \text{also}\ (k\overline{i_1i_2})=0.$$
Hence k is one of the points in which
$$\overline{i_1i_2}\ \text{intersects the curve}\ (\overline{xf}\;\overline{\overline{xg_2}\gamma g_1}\;\overline{\overline{xh_2}\gamma h_1})=0.$$
We consider this curve
$$(\overline{xf}\;\overline{\overline{xg_2}\gamma g_1}\;\overline{\overline{xh_2}\gamma h_1})=0$$
Put $x=\overline{\gamma\xi}$, where ξ is any line.

Then
$$(\overline{\gamma\xi f}\;\overline{\overline{\overline{\gamma\xi}g_2}\gamma g_1}\;\overline{\overline{\overline{\gamma\xi}h_2}\gamma h_1})=(\overline{\gamma\xi f}\;\overline{\gamma\xi g_1}\;\overline{\gamma\xi h_1})=0.$$
Again, put $x=\overline{\beta\xi}$. Now g_2, h_2, f lie on β,
$$\therefore\ \overline{xf}=\beta,\ \overline{\overline{xg_2}\gamma g_1}=\overline{\beta\gamma g_1},\ \overline{\overline{xh_2}\gamma h_1}=\overline{\beta\gamma h_1}.$$
Hence β, γ are parts of the curve.

Hence the remainder of the locus is another line.

To find this line we have

$$(\overline{xf}\ \overline{\overline{xg_2}\gamma g_1}\ \overline{x\overline{h_2}\gamma h_1})=0,$$

$$\therefore\ (\overline{xf}\ \overline{\overline{xg_2}\gamma g_1}\ h_1\gamma h_2 x)=0.$$

This is satisfied if

$$\overline{xf}\ \overline{\overline{xg_2}\gamma g_1}=h_1,$$

$$\text{i.e. } (xfh_1)=0,\ (xg_2\gamma g_1 h_1)=0,$$

$$\text{i.e. } (xfh_1)=0 \text{ and } (h_1 g_1\gamma g_2 x)=0,$$

$$\text{i.e. } x=\overline{fh_1}\ \overline{\overline{h_1 g_1}\gamma g_2}.$$

Similarly another value is given by

$$x=\overline{fg_1}\ \overline{g_1 h_1\gamma h_2},$$

therefore the third line is

$$\overline{\overline{\overline{h_1 g_1}\gamma g_2}\ \overline{fh_1}}\ \overline{\overline{g_1 h_1}\gamma h_2}\ \overline{fg_1}.$$

Denote this, for brevity, by λ.

Then k is incident in $\overline{i_1 i_2}$ and in β or γ, or λ.

If we assume that k lies in β, the equation of the cubic becomes

$$(\overline{\overline{xa}a}\ a_1\ \beta\gamma\ \overline{b_1}\ xc)=0,$$

i.e. a conic and a line.

Similarly if k lies in γ, the cubic becomes

$$(\overline{\overline{xa}a}\ \overline{a_1}\ \overline{kb_1}\ xc)=0,$$

i.e. a conic and a line.

The only possibility then is k lies in λ. It will be shown that this assumption allows the cubic to be of the general type. We shall prove this by showing that the cubic passes through the nine arbitrarily assumed points.

Hence let it be assumed that

$$k=\overline{\overline{i_1 i_2}\lambda}.$$

Accordingly with these assumptions the equations

$$(g_1 b_1\gamma k g_2)=0,\ (h_1 b_1\gamma k h_2)=0,\ (i_1 b_1 c k i_2)=0$$

are satisfied and therefore g, h, i lie on the curve.

b_1 is the point of intersection of

$$\overline{kf},\ \overline{kg_2\gamma g_1},\ \overline{\overline{kh_2}\gamma h_1},$$

$$\therefore\ b_1=\overline{kg_2\gamma g_1}\ \overline{kf}.$$

Finally therefore it has been proved that the cubic curve

$$(\overline{xa}\ a\ \overline{a_1}\ \overline{\overline{\overline{xb}\beta k}\gamma b_1}\ \overline{xc})=0$$

passes through the nine arbitrarily chosen points a, b, c d, e, f g, h, i provided that α, β, γ, a_1, b_1, k are determined by the linear constructions

$$\alpha=\overline{de},\ \beta=\overline{ef},\ \gamma=\overline{ei_1},$$

$$a_1=\overline{\overline{af}\ \overline{cd}},\ k=\overline{\overline{i_1 i_2}\lambda},\ b_1=\overline{\overline{\overline{kg_2}\gamma g_1}\ \overline{kf}},$$

where
$$g_1=\overline{\overline{\overline{g\alpha}\ \overline{de}}\ \overline{\overline{af}\ \overline{cd}}\ \overline{gc}},\quad h_1=\overline{\overline{\overline{h\alpha}\ \overline{de}}\ \overline{\overline{af}\ \overline{cd}}\ \overline{hc}},$$

$$g_2=\overline{\overline{gb}\ \overline{ef}},\qquad h_2=\overline{\overline{hb}\ \overline{ef}},$$

$$i_1=\overline{\overline{\overline{i\alpha}\ \overline{de}}\ \overline{\overline{af}\ \overline{cd}}\ \overline{ic}},$$

$$i_2=\overline{\overline{ib}\ \overline{ef}},$$

$$\lambda=\overline{\overline{\overline{h_1 g_1}\gamma g_2}\ \overline{fh_1}}\ \overline{\overline{\overline{g_1 h_1}\gamma h_2}\ \overline{fg_1}}.$$

This gives us the analogue of Pascal's theorem for a cubic. This theorem and analysis are due to Grassmann.

47. In bi-radial co-ordinates, shew that Laplace's equation is

$$\left(\frac{\partial^2}{\partial r^2}+\frac{\partial^2}{\partial s^2}+2\cos\theta\,\frac{\partial^2}{\partial r\partial s}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{s}\frac{\partial}{\partial s}\right)V=0,$$

where
$$r=|(xa)|,\ s=|(xb)|,\ \theta=(\overline{xa}\,\overline{xb}),$$

a, b being the points of reference.

Use the theorem on p. 77.

48. Find the condition that y is a double point of the curve

$$f\{|(xa)|,\ |(xb)|,\ |(xc)|\dots(x\alpha),\ (x\beta),\ (x\gamma)\dots\}=0,$$

and that η is a double tangent of the curve

$$f\{(\xi a),\ (\xi b),\ (\xi c)\dots(\xi\alpha),\ (\xi\beta),\ (\xi\gamma)\dots\}=0.$$

49. Let $\left.\begin{matrix}a,\ b,\ c\\ \alpha,\ \beta,\ \gamma\end{matrix}\right\}$; $\left.\begin{matrix}a',\ b',\ c'\\ \alpha',\beta',\gamma'\end{matrix}\right\}$ be two triangles and s any point. Shew that

$$\frac{(\overline{sa\alpha'}\ \overline{sb\beta'}\ \overline{sc\gamma'})(sa\alpha')(sb\beta')(sc\gamma')}{(\overline{sa'\alpha}\ \overline{sb'\beta}\ \overline{sc'\gamma})(sa'\alpha)(sb'\beta)(sc'\gamma)}=-\frac{R(a,\ b,\ c)}{R(a',\ b',\ c')},$$

and prove the reciprocal theorem.

On account of the comparative simplicity of the properties of the circle, and the testimony of Pure Geometry, we are warranted to try to include the circle as an element. This is what is done in Euclid. There are three elements, points, lines, circles. The inclusion of the circle is theoretically quite simple and would be analogous to what we have done in regard to the point and line. Let us denote circles by Capital Greek letters, Γ, Λ, etc.

As the radius of a circle is intrinsic to it, we shall express it in the notation. Thus $\underset{a}{\Gamma}$, $\underset{b}{\Gamma}$ are circles with radius a, b.

Now the position of a point in regard to a circle is completely given if we know, say, the distance of the point from its centre or the length of the tangent from the point to the circle. Experience shews that this latter is

the most convenient to work with. We shall denote therefore the length of the tangent by

$$(a \underset{r}{\Gamma}) \text{ or } (\underset{r}{\Gamma} a).$$

In regard to the measure of a line and circle we choose the measure of the centre of circle and line, divided by the radius. When the line cuts the circle this becomes the cosine of the angle between the line and circle. It is essential that the circle be given a sense, as in the case of a line. We shall suppose that the radius of a circle is positive when the sense is counter-clockwise and negative when clockwise.

Our notation would be $(\alpha \underset{r}{\Gamma})$ or $(\underset{r}{\Gamma} \alpha)$.

We have $$(\bar{\alpha} \underset{r}{\Gamma}) = (\alpha \underset{r}{\bar{\Gamma}}) = -(\alpha \underset{r}{\Gamma}).$$

When the line and circle intersect we shall use $(\widehat{\alpha \underset{r}{\Gamma}})$ to denote the angle between the line and circle.

In regard to the measure of two circles of given radius, their mutual position is given by the distance of their centres. We shall however define the measure of two circles $\underset{a}{\Gamma}, \underset{b}{\Lambda}$ by

$$2ab\,(\underset{a}{\Gamma}\,\underset{b}{\Lambda}) = (c'\underset{a}{\Gamma}\,c'\underset{b}{\Lambda})^2 - a^2 - b^2,$$

where $c'\underset{a}{\Gamma}$ denotes the centre of $\underset{a}{\Gamma}$.

The quantity on the right-hand side is usually called the power of the two circles

When the circles cut $(\underset{a}{\Gamma}\,\underset{b}{\Lambda})$ is the cosine of the angle between the circles.

The analogue of determinates of points and lines is a much more detailed matter.

For a point and circle we have $\overline{a\underset{r}{\Gamma}}$, the pairs of tangents from a to $\underset{r}{\Gamma}$.

The determinate of a line and circle $\overline{\Gamma\alpha}$ is the pair of points of intersection.

When we come to the consideration of two circles we find we have more than one determinate which leads to considerable detail. We shall not go into this any further. The inclusion of the circle as a third element would mean a table of formulae over ten times as large as the table given. We give a few examples of this theory.

50. The circle whose centre is o and radius r we shall write cir $\overline{o, r}$: if a be a point on the circle, shew that the intercept on a_ω is

$$2r \cos(\overline{ao}\,\omega).$$

Let l, λ be a point and line; the locus of x such that $(xl)^2 = 2k\,(x\lambda)$ is a circle. The circle we shall write cir $\overline{l, \lambda; k}$. If a be a point on the cir $\overline{l, \lambda; k}$, shew that the intercept on the line a_ω is

$$2\,|(al)|\cos(\overline{al}\,\omega) - 2k\sin(\omega\lambda).$$

51. *M'Cay's Theorem.* If a, b be two points on cir $\overline{l, \lambda; k}$, cir $\overline{l, \lambda; k'}$ respectively, and such that $(\overline{la}\,\overline{lb}) = \frac{\pi}{2}$, shew that the intercepts on $\overline{ab}$ made by the two circles are as $k : k'$.

We have
$$(al)^2 = 2k\,(a\lambda) \quad \text{..(i)},$$
$$(bl)^2 = 2k'\,(b\lambda) \quad \text{..(ii)},$$
$$(\overline{la}\,\overline{lb}) = \frac{\pi}{2} \quad \text{..(iii)}.$$

These are the conditions of the problem.

Using the result of Ex. 50 we are required to shew that
$$\frac{2\,|(al)|\cos(\overline{al}\,\overline{ab}) - 2k\sin(\overline{ab}\lambda)}{2\,|(bl)|\cos(\overline{bl}\,\overline{ba}) - 2k'\sin(\overline{ba}\lambda)} = \frac{k}{k'},$$

i.e. $\quad -k'\,|(la)\,(ab)|\cos(\overline{la}\,\overline{ab}) - 2kk'\,(ab\lambda) + k\,|(lb)\,(ba)|\cos(\overline{lb}\,\overline{ba}) = 0 \ldots (\text{A}).$

Here λ only occurs as a direction. Hence from (i) and (ii) we get
$$k'\,(al)^2 - k\,(bl)^2 = 2kk'\,(ab\lambda).$$

This reduces the L.H.S. of (A) to

$k'\{(la)^2 + (ab)^2 - (bl)^2\} - 2k'\,(al)^2 + 2k\,(bl)^2 + k\{(al)^2 - (lb)^2 - (ba)^2\}$ which is $= 0$,

since
$$(al)^2 + (bl)^2 = (ab)^2.$$

52. If a, b, $\underset{r}{\Gamma}$ be two points and a circle, shew that
$$\sin^2(\overset{<}{\overline{ab}\,\underset{r}{\Gamma}}) = \frac{(a\underset{r}{\Gamma})^4 + (b\underset{r}{\Gamma})^4 + (ab)^4 - 2\,(b\underset{r}{\Gamma})^2\,(ab)^2 - 2\,(ab)^2\,(a\underset{r}{\Gamma})^2 - 2\,(a\underset{r}{\Gamma})^2\,(b\underset{r}{\Gamma})^2}{4\,(ab)^2\,r^2}.$$

53. If a be a point on the circle cir $\overline{l, k; \underset{r}{\Gamma}}$, that is, the circle $(xl)^2 = k^2\,(x\underset{r}{\Gamma})^2$, shew that the intercept on a_ω is
$$\frac{2}{1-k^2}\left[\,|(al)|\cos(\overline{al}\,\omega) - \left|\sqrt{(a\underset{r}{\Gamma})^2 + r^2\sin^2(\overset{<}{a_\omega\,\underset{r}{\Gamma}})}\right|\,\right].$$

54. Hence prove M'Cay's theorem of Ex. 51.

55. Shew that
$$\sin^2(\alpha\beta)\,(\overline{\alpha\beta}\,\underset{r}{\Gamma})^2 = r^2\,\{1 - \cos^2(\alpha\beta) - \cos^2(\alpha\underset{r}{\Gamma}) - \cos^2(\beta\underset{r}{\Gamma})$$
$$+ 2\cos(\alpha\beta)\cos(\alpha\underset{r}{\Gamma})\cos(\beta\underset{r}{\Gamma})\}.$$

56. Shew that $\quad \sum\limits_{\alpha,\beta,\gamma} (\beta\gamma\delta)\sin(\alpha\epsilon) = (\alpha\beta\gamma)\sin(\delta\epsilon).$

57. Let $\left.\begin{matrix} a, b, c \\ \alpha, \beta, \gamma \end{matrix}\right\}, \left.\begin{matrix} a', b', c' \\ \alpha', \beta', \gamma' \end{matrix}\right\}$ be two triangles: if λ_1, μ_1, ν_1 be lines through a, b, c making angle θ with α', β', γ' and λ_2, μ_2, ν_2 lines through a', b', c' making angle $-\theta$ with α, β, γ, then
$$\frac{(\lambda_1\mu_1\nu_1)}{(\lambda_2\mu_2\nu_2)} = \frac{R\,(a, b, c)}{R\,(a', b', c')}.$$

If the lines λ_1, μ_1, ν_1 and therefore λ_2, μ_2, ν_2 are concurrent for the values 0 and $\frac{\pi}{2}$ of θ, shew that they are concurrent for any value of θ.

58. *Directly similar figures: Definition.* Directly similar figures are such that corresponding angles are equal.

If a, a' be two corresponding points, shew that a' may be expressed in the form

$$a' = a_{\overline{oa}_\theta,\, k\,|(oa)|}.$$

o is called the pole, k the scale of similarity and θ the angle of displacement.

59. *Inversely similar figures: Definition.* Inversely similar figures are such that corresponding angles are equal in magnitude but opposite in sign.

If a, a' are two corresponding points, shew that a' may be expressed as

$$a' = o_{\omega_{-(\omega\overline{oa})},\, k\,|(oa)|}.$$

o is called the pole, o_ω the axis and k the scale of transformation.

60. Shew that

$$(a_{\overline{ab}_\theta}\, b_{\overline{ab}_\phi}\, l)^2 \sin^2(\theta - \phi) = (la)^2 \sin^2\theta + (lb)^2 \sin^2\phi$$
$$- 2\,|(la)(lb)|\sin\theta \sin\phi \cos\{(\overline{la}\,\overline{lb}) + \theta - \phi\}.$$

61. If two triangles be inversely similar, shew that they are such that lines through the vertices of one making a constant angle with the sides of the other, are concurrent: and conversely.

62. If two figures are directly similar; and the angle of displacement be a right angle · shew that lines through the vertices of one parallel to the sides of the other are concurrent.

63. If two pairs of triangles $\left.\begin{matrix} a_1 b_1 c_1 \\ \alpha_1\beta_1\gamma_1 \end{matrix}\right\}$, $\left.\begin{matrix} a_2 b_2 c_2 \\ \alpha_2\beta_2\gamma_2 \end{matrix}\right\}$; $\left.\begin{matrix} a_3 b_3 c_3 \\ \alpha_3\beta_3\gamma_3 \end{matrix}\right\}$, $\left.\begin{matrix} a_4 b_4 c_4 \\ \alpha_4\beta_4\gamma_4 \end{matrix}\right\}$ be inversely similar and have a common axis of similitude, shew that if the triangles, suffix 1 and 4, are such that lines through one making an angle θ with the sides of the other are concurrent, then the triangles, suffix 2 and 3, are such that the lines through the vertices of one making an angle θ with the sides of the other are also concurrent.

64. A circle touches three consecutive positions of a moving circle: find its centre and radius.

If the moving circle be cir $\overline{o, r}$ and the touching circle cir $\overline{a, \rho}$ then

$$|(oa)| = \pm(r - \rho),$$

and this can be differentiated twice with a fixed and ρ constant.

65. Find $\left(\frac{d}{d\xi}\right)^r \log(\xi a)$, $r = 1, 2, 3, 4$.

$$\frac{d}{d\xi}\log(\xi a) = \frac{(\nu\xi a)}{(\xi a)},$$

$$\left(\frac{d}{d\xi}\right)^2 \log(\xi a) = \frac{\rho\xi\,(\xi a) - (p\xi a)^2}{(\xi a)^2},$$

$$\left(\frac{d}{d\xi}\right)^3 \log(\xi a) = \frac{\rho'\xi\,(\xi a)^2 - 3\rho\xi\,(\xi a)\,(\nu\xi a) + 2\,(\nu\xi a)\,(p\xi a)^2}{(\xi a)^3},$$

$$\begin{aligned}\left(\frac{d}{d\xi}\right)^4 \log(\xi a) = [(\rho''\xi + 5\rho\xi)\,(\xi a)^3 - 4\rho'\xi\,(\xi a)^2\,(\nu\xi a)\\ + 12\rho\xi\,(\nu\xi a)^2\,(\xi a) - 3\,(\rho\xi)^2\,(\xi a)^2\\ - 2\,(p\xi a)^2\,\{(\xi a)^2 + 3\,(\nu\xi a)^2\}]/(\xi a)^4.\end{aligned}$$

66. Two points a, b move on two lines c_α, c_β in such a manner that $|(ab)|$ is constant. Shew that, if q be the intersection of $\overline{ab}$ with its consecutive position, q and the foot of the perpendicular from c on $\overline{ab}$ are isotomic conjugates in regard to a, b. [*The Principia.*]

We have to shew that

$$(\nu\,\overline{ab}\,a) = (\overline{\tau a\,\tau b}\;b_{\overline{ab}_{\frac{\pi}{2}}}),$$

when

$$|(ab)| = \text{constant}.$$

Now

$$\begin{aligned}(\nu\,\overline{ab}\,a) &= \frac{|(ab)|\,(\tau ab)\,da}{(\tau ab)\,da + (\tau ba)\,db}\\ &= \frac{|(ab)|\,(\tau ab)\,(\nu ba)}{(\tau ab)\,(\nu ba) - (\tau ba)\,(\nu ab)},\end{aligned}$$

since $(\nu ab)\,da + (\nu ba)\,db = 0$,

$$\begin{aligned}&= \frac{(\tau ab)\sin(\tau b\,\overline{ab}_{\frac{\pi}{2}})}{\sin(\tau a\tau b)}\\ &= (\overline{\tau a\tau b}\;b_{\overline{ab}_{\frac{\pi}{2}}}).\end{aligned}$$

67. Shew that the conic which has three-point contact with a curve at the point a and has a focus at s has the equation

$$2\,|(xs)\,(as)| - (as)^2 - (sx)^2 + (ax)^2 = 2\rho a\,\frac{(\tau as)^2}{(as)^2}\,(x\tau a).$$

68. By means of Ex. 67 or otherwise find an equation of the locus of the foci of conics having four-point contact at a point of a curve.

69. Shew that

$$\begin{aligned}\int(\xi a)(\xi b)\,d\xi = \tfrac{1}{2}\,(a\xi)\,\{(\xi a)\,(\xi b) + (\nu\xi a)\,(\nu\xi b)\}\\ - \tfrac{1}{4}\,\{(\xi a)\,(\nu\xi b) + (\xi b)\,(\nu\xi a)\}\\ + \{(\xi a) + (\xi b)\}\,\{-\tfrac{1}{2}\,(a\xi)\,(\rho\xi - \rho''\xi + \rho^{\text{iv}}\xi - \dots)\\ + (1 + \tfrac{1}{4})\,\rho'\xi - (2 + \tfrac{1}{4})\,\rho'''\xi + (3 + \tfrac{1}{4})\,\rho^{\text{v}}\xi - \dots\}\\ + \{(\nu\xi a) + (\nu\xi b)\}\,\{-\tfrac{1}{2}\,(a\xi)\,(\rho'\xi - \rho'''\xi + \rho^{\text{v}}\xi - \dots)\\ - \tfrac{3}{4}\rho\xi + (1 + \tfrac{3}{4})\,\rho''\xi - (2 + \tfrac{3}{4})\,\rho^{\text{iv}}\xi + \dots\},\end{aligned}$$

where a is an arbitrary line.

The infinite series occurring are supposed convergent and differentiable.

Assume
$$\int (\xi a)(\xi b)\,d\xi = I_0 + I_1(\xi a)(\xi b) + I_2(\nu\xi a)(\nu\xi b) + I_3\{(\xi a)(\nu\xi b)+(\xi b)(\nu\xi a)\} + I_4\{(\xi a)+(\xi b)\} + I_5\{(\nu\xi a)+(\nu\xi b)\},$$
where the I's are intrinsic functions of the curve and differentiate.

70. Prove generally
$$\int P\{(\xi a), (\xi b)\ldots(\nu\xi a), (\nu\xi b)\ldots \sin(\xi a), \sin(\xi\beta)\ldots I_1(\xi), I_2(\xi)\ldots\}\,d\xi$$
$$= P\{(\xi a), (\xi b)\ldots(\nu\xi a), (\nu\xi b)\ldots \sin(\xi a), \sin(\xi\beta)\ldots I_1(\xi), I_2(\xi)\ldots\},$$
where P denotes a polynomial, and the I's denote intrinsic functions.

71. In the cubic
$$(xa\beta c\delta x\delta_1 c_1\beta_1 a_1 x)=0,$$
find where δ, δ_1; $\overline{ca}$, $\overline{c_1a_1}$ cut the curve and shew that β cuts the curve in the points where it cuts the conic $(xc\delta_1 c_1\beta_1 a_1 x)=0$.

To find where δ cuts the curve, put $x=\overline{\delta\zeta}$, then $(\zeta\delta a\beta c\delta\overline{\zeta\delta}\,\delta_1 c_1\beta_1 a_1\delta\zeta)=0$.

Hence $\overline{\overline{\overline{\overline{\zeta\delta}\,a}\,\beta}\,c}\,\delta=\overline{\zeta\delta}\ldots$(i) or $(\delta\delta_1 c_1\beta_1 a\zeta\delta)=0\ldots$(ii).

From (i)
$$(\zeta\delta a\beta c\zeta\delta)=0,$$
$$\therefore\ \zeta\delta=\overline{\beta\delta}\ \text{ or }\ \overline{\overline{ca}\,\delta}.$$

From (ii)
$$\overline{\zeta\delta}=\overline{\overline{\overline{\overline{\delta\delta_1}\,c_1}\,\beta_1}\,a}\,\delta.$$

Hence the three points of intersection are
$$\overline{\beta\delta},\ \overline{\overline{ca}\,\delta},\ \overline{\overline{\overline{\overline{\delta\delta_1}\,c_1}\,\beta_1}\,a}\,\delta.$$

To find where $\overline{ca}$ cuts the curve, put $x=\overline{\overline{ca}\,\zeta}$. And we find in a similar manner that the points of intersection are
$$a,\ \overline{\overline{ca}\,\delta},\ \overline{\overline{\overline{\overline{\overline{ca}\,\delta_1}\,c_1}\,\beta_1}\,a_1}\ \overline{ca}.$$

72. In the cubic $(\overline{\overline{\overline{\overline{\overline{xa}\,\beta}\,c}\,\delta}\,e}\,\theta\ \overline{\overline{xl}\,\mu}\ \overline{\overline{xr}\,\sigma})=0$,
shew that a, l, r, $\overline{\mu\sigma}$, $\overline{\sigma\theta}$ lie on the curve. And find the third points in which $\overline{al}$, $\overline{lr}$, $\overline{\overline{\mu\sigma}\,r}$, $\overline{\overline{\mu\sigma}\,a}$ cut the curve.

73. *Notation.* We shall denote by a_k the line parallel to a and such that the measure of any point on a_k and a is k.

It is easy to shew that $(a_k b)=(ab)-k$
and evidently $(a_k\beta)=(a\beta)$.

74. Let a, b, c; α, β, γ be three points and three lines. If $\left.\begin{matrix} x, & y, & z \\ \xi, & \eta, & \zeta \end{matrix}\right\}$ be a triangle such that the points are incident in α, β, γ and the lines in a, b, c: shew how to find $(x\lambda)$ where λ is any line.

We have
$$(x\alpha) = (y\beta) = (z\gamma) = 0,$$
$$(yza) = (zxb) = (xyc) = 0,$$
we are required to find $(x\lambda)$.

We may put $x = \overline{\lambda_k \alpha}$, if we suppose $(x\lambda) = k$.

Hence
$$y = \overline{\overline{xc}\beta} = \overline{\overline{\overline{\lambda_k \alpha}\, c}\, \beta},$$
$$z = \overline{\overline{xb}\gamma} = \overline{\overline{\overline{\lambda_k \alpha}\, b}\, \gamma}.$$

Hence
$$(\overline{\overline{\overline{\lambda_k \alpha}\, c}\, \beta}\ \ \overline{\overline{\overline{\lambda_k \alpha}\, b}\, \gamma}\ \ a) = 0.$$

From which it is easy to shew that
$$(\gamma\alpha\lambda_k)(cb\alpha\lambda_k)(a\beta) + (ac\alpha\lambda_k)(\beta\gamma b\alpha\lambda_k) = 0,$$
$$\therefore\ \{(\gamma\alpha\lambda) - k\sin(\gamma\alpha)\}\{(cb\alpha\lambda) - k(cb\alpha)\}(a\beta)$$
$$+ \{(ac\alpha\lambda) - k(ac\alpha)\}\{(\beta\gamma b\alpha\lambda) - k(\beta\gamma b\alpha)\} = 0,$$
$$\therefore\ (\gamma\alpha\lambda)(cb\alpha\lambda)(\beta a) + (ac\alpha\lambda)(\beta\gamma b\alpha\lambda)$$
$$- k[(\gamma\alpha\lambda)(cb\alpha)(a\beta) + \sin(\gamma\alpha)(cb\alpha\lambda)(a\beta)$$
$$+ (ac\alpha\lambda)(\beta\gamma b\alpha) + (ac\alpha)(\beta\gamma b\alpha\lambda)]$$
$$+ k^2[\sin(\gamma\alpha)(cb\alpha)(a\beta) + (ac\alpha)(\beta\gamma b\alpha)] = 0.$$

75. If in Ex. 74
$$(\beta\gamma bc) = (\gamma\alpha ca) = (\alpha\beta ab) = 0,$$
shew that one solution for x, y, z is $\overline{bc\alpha}$, $\overline{\overline{ca}\beta}$, $\overline{\overline{ab}\gamma}$ and find the other solution.

Also consider the case in which
$$(abc) = (\alpha\beta\gamma) = 0.$$

76. If $\quad a(f_\alpha g_\eta h_\zeta) = b(f_\xi g_\beta h_\zeta) = c(f_\xi g_\eta h_\gamma) = (f_\xi g_\eta h_\zeta)$,
shew that $(f_\xi g_\eta h_\zeta)$ satisfies a cubic in measures of $f, g, h, \alpha, \beta, \gamma$ and a, b, c.

77. Shew that for a cubic curve if $\left.\begin{matrix} a, & b, & c \\ \alpha, & \beta, & \gamma \end{matrix}\right\}$, $\left.\begin{matrix} a', & b', & c' \\ \alpha', & \beta', & \gamma' \end{matrix}\right\}$ be two triangles such that their vertices and the intersections of corresponding sides lie on the cubic, then
$$(\alpha'\beta'\gamma') = l(\alpha\beta'\gamma') = m(\alpha'\beta\gamma') = n(\alpha'\beta'\gamma),$$
and hence shew that the cubic is not restricted by such a condition.

Hence shew that the cubic
$$(\overline{\overline{xa}\alpha}\ \ \overline{\overline{xb}\beta}\ \ \overline{\overline{xc}\gamma}) = 0$$
is a *general* cubic*.

* Due to Grassmann.

78. Find the equation of a circle, the measures of which in regard to the three points a, b, c are t_a, t_b, t_c.

Let x, y be points on the circle. Let τy be the tangent at y.

Then $(xy)^2 = 2\rho\,(x\tau y)$, where ρ is the radius.

It is easy to shew that

$$t_a^2 = (ay)^2 - 2\rho\,(a\tau y),$$
$$t_b^2 = (by)^2 - 2\rho\,(b\tau y),$$
$$t_c^2 = (cy)^2 - 2\rho\,(c\tau y).$$

If we eliminate τy we find the equation of the circle.

Now from a, b, c, y, τy we have

$$\Sigma\,(\tau y a)\,(bcy) = 0,$$
$$\therefore\ \Sigma\,\{t_a^2 - (ay)^2\}\,(bcy) = 0,$$
$$\therefore\ (abc)\,t^2 + \Sigma\,(bcy)\,t_a^2 = 0,$$

where t is the measure of y in regard to the circumcircle of a, b, c.

If we denote by cir $\overline{a,\,b,\,c}$ the circumcircle of a, b, c the equation of a circle Γ may be written in the form

$$(x \operatorname{cir} \overline{a,\,b,\,c})^2\,(abc) + \sum_{a,\,b,\,c} (a\Gamma)^2\,(bcx) = 0.$$

79. Find the radius of a circle Γ when $(\Gamma\alpha)$, $(\Gamma\beta)$, $(\Gamma\gamma)$ are given.

We have $$(o\alpha) = r\cos(\Gamma\alpha),$$

and two other equations.

$$\therefore\ r\Sigma\sin(\beta\gamma)\cos(\Gamma\alpha) = (\alpha\beta\gamma).$$

80. Transform cir $\overline{l,\lambda\,;k}$ to the form cir $\overline{o,\,r}$.

$$\operatorname{cir}\overline{l,\lambda\,;k} \text{ is equivalent to } \operatorname{cir}\overline{l_{\lambda_{\frac{\pi}{2}},\,-k},\ \sqrt{k^2+2k\,(l\lambda)}}.$$

81. If we denote by ra $\overline{\Gamma_1\Gamma_2}$ the radical axis of the circles Γ_1, Γ_2, shew that

$$|(\overline{\text{ra}\operatorname{cir}\overline{o_1,\,r_1}\operatorname{cir}\overline{o_2,\,r_2}}\,x)| = \left|\frac{(xo_1)^2 - (xo_2)^2 - r_1^2 + r_2^2}{2\,(o_1o_2)}\right|.$$

82. If a circle touch two circles, shew that the perpendicular from its centre on the radical axis of the two circles is proportional to its radius.

Let the touching circle be cir $\overline{o,\,r}$ and the other circles be cir $\overline{o_1,\,r_1}$ and cir $\overline{o_2,\,r_2}$.

We have

$$|(\overline{\text{ra}\operatorname{cir}\overline{o_1,\,r_1}\operatorname{cir}\overline{o_2,\,r_2}}\,o)|/r = \left|\frac{(oo_1)^2 - (oo_2)^2 - r_1^2 + r_2^2}{2\,(o_1o_2)\,r}\right|$$
$$= \left|\frac{(r-r_1)^2 - (r-r_2)^2 - r_1^2 + r_2^2}{2\,(o_1o_2)\,r}\right|,$$

since $$(r-r_1)^2 = (oo_1)^2,\ (r-r_2)^2 = (oo_2)^2,$$
$$= \left|\frac{r_1 \pm r_2}{(o_1o_2)}\right| \text{ a constant.}$$

83. Find the condition for a double point at the point h of the curve

$$f\{|(xa_1)|,\ |(xa_2)| \dots (xa_1),\ (xa_2) \dots\}=0.$$

84. Find the condition for a double tangent at the line λ of the curve

$$f\{(\xi a_1),\ (\xi a_2) \dots\ (\xi a_1),\ (\xi a_2) \dots\}=0.$$

85. Find the tangent and radius of curvature of the curve given by

$$\phi\{|(xa_1)|,\ |(xa_2)| \dots (xa_1),\ (xa_2) \dots t\}=0,$$
$$\psi\{|(xa_1)|,\ |(xa_2)| \dots (xa_1),\ (xa_2) \dots t\}=0,$$

where t is a variable parameter.

86. Defining $(a_1 a_2 a_1 a_3 a_2 \dots a_{n-1} a_n)$ as

$$|(a_1 a_2)| \sin(\overline{a_1 a_2} a)\, |(\overline{\overline{a_1 a_2} a_1 a_3})| \sin(\overline{\overline{\overline{a_1 a_2} a_1 a_3} a_2}) \dots (\overline{\overline{\overline{a_1 a_2} a_1 a_3} a_2 \dots a_{n-1} a_n}),$$

shew that

$$(a_1 a_2 a_1 a_3 a_2 \dots a_{n-1} a_n)=\begin{vmatrix} (a_1a_1) & (a_1a_2) & (a_1a_3) \dots (a_1a_{n-1}) & (a_1a_n) \\ (a_2a_1) & (a_2a_2) & (a_2a_3) \dots (a_2a_{n-1}) & (a_2a_n) \\ 0 & (a_3a_2) & (a_3a_3) \dots (a_3a_{n-1}) & (a_3a_n) \\ \cdot & \cdot & \cdot \quad \dots \quad \cdot & \cdot \\ \cdot & \cdot & \cdot \quad \dots \quad \cdot & \cdot \\ \cdot & \cdot & \cdot \quad \dots \quad \cdot & \cdot \\ 0 & 0 & 0 \quad \dots (a_na_{n-1}) & (a_na_n) \end{vmatrix}.$$

87. Shew that

$$(a_1 a_2 a_1 a_3 a_2 a_4 \dots a_{n-1} a_{n-2} a_n)$$
$$=\begin{vmatrix} (a_1a_1) & (a_1a_2) & (a_1a_3) \dots (a_1a_{n-2}) & \sin(a_1a_n) \\ (a_2a_1) & (a_2a_2) & (a_2a_3) \dots (a_2a_{n-2}) & \sin(a_2a_n) \\ 0 & (a_3a_2) & (a_3a_3) \dots (a_3a_{n-2}) & \sin(a_3a_n) \\ \cdot & \cdot & \cdot \quad \dots \quad \cdot & \cdot \\ \cdot & \cdot & \cdot \quad \dots \quad \cdot & \cdot \\ \cdot & \cdot & \cdot \quad \dots \quad \cdot & \cdot \\ 0 & 0 & 0 \quad \dots (a_{n-1}a_{n-2}) & \sin(a_{n-1}a_n) \end{vmatrix}.$$

88. Let $\left.\begin{matrix} a, b, c \\ \alpha, \beta, \gamma \end{matrix}\right\}$ be a triangle. A circle cuts the sides α, β, γ in the points a_1, a_2; b_1, b_2; c_1, c_2 respectively, such that $\overline{b_2 c_1}$, $\overline{c_2 a_1}$ are parallel to given directions. Shew that the locus of the centre of such circles is a straight line.

89. Find ρx_ω.

We have $$(px_\omega a)=(xa)-\sin(\omega a)\sin(\tau x\omega)\frac{dx}{d\omega}.$$

Hence differentiating

$$-\sin(\omega a)\,\rho x_\omega = -\sin(\tau x a)\frac{dx}{d\omega}+\cos(\omega a)\sin(\tau x\omega)\frac{dx}{d\omega}$$
$$-\sin(\omega a)\cos(\tau x\omega)\frac{dx}{d\omega}\left(1-\rho x\frac{dx}{d\omega}\right)$$
$$-\sin(\omega a)\sin(\tau x\omega)\frac{d^2x}{d\omega^2}$$

$$= -\cos(\tau x\omega)\sin(\omega a)\frac{dx}{d\omega} - \sin(\omega a)\cos(\tau x\omega)\frac{dx}{d\omega}\left(1 - \rho x\frac{dx}{d\omega}\right)$$
$$- \sin(\omega a)\sin(\tau x\omega)\frac{d^2x}{d\omega^2}.$$

Hence $$\rho x_\omega = \cos(\tau x\omega)\left\{2\frac{dx}{d\omega} - \rho x\left(\frac{dx}{d\omega}\right)^2\right\} + \sin(\tau x\omega)\frac{d^2x}{d\omega^2}.$$

90. When x has the general displacement, shew that
$$\rho x_\omega = 2\sum_r \cos(\tau_r x\omega)\frac{d_r x}{d\omega} - \sum_r \rho_r x\left(\frac{d_r x}{d\omega}\right)^2 \cos(\tau_r x\omega) + \sum_r \sin(\tau_r x\omega)\frac{d_r{}^2 x}{d\omega^2}.$$

91. pq is the chord of a continuous curve cutting off an arc of constant length; the tangents at p, q meet in t, the bisector of the lines $\overline{pt}$, qt meets $\overline{pq}$ in r; if r' be the isotomic conjugate of r in regard to p, q, prove that r' is the intersection of $\overline{pq}$ with its consecutive position.

92. If x and y be points such that $x\bar{y} = \tau x$, then
$$d\,|(xy)| = dx - \cos(\tau x\tau y)\,dy.$$

If x_1, x_2 be the points of contact of two tangents from y to a curve, then
$$d\sigma = dx_1 - dx_2 - \{\cos(\tau x_1\tau y) - \cos(\tau x_2\tau y)\}\,dy,$$
where σ is the sum of the lengths of the two tangents from y to the curve.

Let us consider an ellipse.

If we suppose $d\sigma = dx_1 - dx_2$, then
$$\cos(\tau x_1\tau y) = \cos(\tau x_2\tau y),$$
$$\therefore\ (\tau x_1\tau y) + (\tau x_2\tau y) = 0.$$

Now if we use the theorem that the tangents from a point to an ellipse are isogonal conjugates in regard to the joins of the point with the foci, we have
$$(\tau y\,\overline{ys_1}) + (\tau y\,\overline{ys_2}) = 0.$$

Hence by integrating, the locus of y is a confocal ellipse.

The integration of $d\sigma = dx_1 - dx_2$ is that the sum of the lengths of the tangents exceeds the length of the intercepted arc by a constant quantity.

This is *Grave's Theorem*, viz., that the sum of the lengths of the tangents from a point on an ellipse to a confocal ellipse exceeds the length of the intercepted arc by a constant quantity.

93. Shew that $$\rho\lambda_k = \rho\lambda - k - \frac{d^2k}{d\lambda^2}.$$

94. Shew that

$$d\tau x_{\hat\rho\hat\sigma} \quad \dot\omega\,(dx_{\hat\rho\hat\sigma}\,.\,\omega)^2$$
$$= (dx)^2\,d\tau x + d\tau x\,dx\sum_{\hat\rho}\{\hat\rho\cos(\tau x\rho) - \hat\rho\,d\rho\sin(\tau x\rho)\}$$
$$- d^2x\sum_\rho\{\hat\rho\,d\rho\cos(\tau x\rho) + d\hat\rho\sin(\tau x\rho)\}$$
$$+ dx\sum_\rho\{\cos(\tau x\rho)(2d\hat\rho\,d\rho + \hat\rho\,d^2\rho) + \sin(\tau x\rho)(d^2\hat\rho - \hat\rho\,(d\rho)^2\}$$
$$+ \sum_\rho\{-d^2\hat\rho\hat\rho\,d\rho + 2(d\hat\rho)^2\,d\rho + \hat\rho\,d^2\rho\,d\hat\rho + \hat\rho^2\,d\hat\rho\,(d\rho)^2\}$$
$$+ \sum_{\rho\neq s}\cos(\rho\sigma)\{-(d^2\hat\rho\hat\sigma\,d\sigma + d^2\hat\sigma\hat\rho\,d\rho) + (d\rho + d\sigma)(2d\hat\rho\,d\hat\sigma + \hat\rho\hat\sigma\,d\rho\,d\sigma)$$
$$+ (\hat\rho\,d^2\rho\,d\sigma + \hat\sigma\,d^2\sigma\,d\rho)\}$$
$$+ \sum_{\rho,\sigma}\{(d\hat\rho\,d^2\hat\sigma - d^2\hat\rho\,d\hat\sigma) + 2d\rho\,d\sigma\,(\hat\rho\,d\hat\sigma - \hat\sigma\,d\hat\rho) + \hat\rho\hat\sigma\,(d\rho\,d^2\sigma - d^2\rho\,d\sigma)$$
$$+ (\hat\rho\,d\hat\sigma\,(d\rho)^2 - \hat\sigma\,d\hat\rho\,(d\sigma)^2)\}.$$

95. Shew that

$$\rho x_{\rho t} \quad \phi_\omega = \sum_\rho \hat{\rho} \sin(\rho\omega) + \cos(\tau x\omega)\left(2 - \frac{d\tau x}{d\omega}\right)\frac{dx}{d\omega}$$
$$+ \sin(\tau x\omega)\frac{d^2x}{d\omega^2} + \sum_\rho \sin(\rho\omega)\left\{\frac{d^2\hat{\rho}}{d\omega^2} - \hat{\rho}\left(1 - \frac{d\rho}{d\omega}\right)^2\right\}$$
$$+ \sum_\rho \cos(\rho\omega)\left\{2\frac{d\hat{\rho}}{d\omega}\left(1 - \frac{d\rho}{d\omega}\right) - \hat{\rho}\frac{d^2\rho}{d\omega^2}\right\}.$$

96. Shew that $$(\overline{a.b}_{1\overline{ab}_{\frac{\pi}{2}}}\, o) = \frac{(oa)^2 - (ob)^2}{-2\,|(ab)|}.$$

97. Shew that

$$\sum_r A_r (xa_r)^2 + 2\sum_r B_r (x\beta_r) + C = 0$$

is a line, when $$\sum_r A_r = 0,$$

and find $$(\sum_r A_r (xa_r)^2 + 2\sum_r B_r (x\beta_r) + C = 0 \;\; d)$$

when $$\sum_r A_r = 0.$$

Subtract $(xc)^2 \sum_r A_r$ from the equation, where c is an arbitrary point, and we have

$$\sum_r A_r \{(xa_r)^2 - (xc)^2\} + 2\sum_r B_r (x\beta_r) + C = 0,$$
$$\therefore \quad -2\sum_r A_r\,|(a_r c)|\,(\overline{a_r.c}_{1\overline{a_r c}_{\frac{\pi}{2}}}\, x) + 2\sum_r B_r (x\beta_r) + C = 0,$$

from Ex. 96 which is a line.

Hence $$(\sum_r A_r (xa_r)^2 + 2\sum_r B_r (x\beta_r) + C = 0 \;\; d)$$
$$= \frac{-2\sum_r A_r\,|(a_r c)|\,(\overline{a_r.c}_{1\overline{a_r c}_{\frac{\pi}{2}}}\, d) + 2\sum_r B_r (d\beta_r) + C}{2\Omega}$$
$$= \frac{\sum_r A_r (da_r)^2 + 2\sum_r B_r (x\beta_r) + C}{2\Omega},$$

where $$\Omega^2 = \sum_r A_r^2 (a_r c)^2 + \sum B_r^2 + 2\sum_{r \neq s} A_r A_s\,|(a_r c)(a_s c)|\cos(\overline{a_r c}\,\overline{a_s c})$$
$$+ 2\sum_{r \neq s} B_r B_s \cos(\beta_r \beta_s)$$
$$+ 2\sum_{r,s} A_r B_s\,|(a_r c)|\sin(\beta_s\,\overline{a_r c})$$
$$= \sum_r A_r^2 (a_r c)^2 + \sum B_r^2 + 2\sum_{r \neq s} A_r A_s\,|(a_r c)(a_s c)|\cos(\overline{a_r c}\,\overline{a_s c})$$
$$+ 2\sum_{r \neq s} B_r B_s \cos(\beta_r \beta_s)$$
$$- 2\sum_{r,s} A_r B_s (a_r c \beta_s)$$
$$= -\sum_{r \neq s} A_r A_s (a_r a_s)^2 + \sum_r B_r^2 + 2\sum_{r \neq s} B_r B_s \cos(\beta_r \beta_s)$$
$$- 2\sum_{r,s} A_r B_s (a_r \beta_s).$$

Hence
$$(\sum_r A_r(xa_r)^2+2\sum_r B_r(x\beta_r)+C=0\ d)$$
$$=\frac{\sum_r A_r(da_r)^2+2\sum_r B_r(d\beta_r)+C}{2\sqrt{-\sum_{r\neq s}A_rA_s(a_ra_s)^2+\sum B_r^2+2\sum_{r\neq s}B_rB_s\cos(\beta_r\beta_s)-2\sum_{r,s}A_rB_s(a_r\beta_s)}}.$$

98. Shew that, when $\sum_r A_r=0$
$$\sin(\sum_r A_r(xa_r)^2+2\sum_r B_r(x\beta_r)+C=0\ \delta)$$
$$=\frac{-\sum A_r(a_r\delta_{\frac{\pi}{2}})+\sum_r B_r\sin(\delta\beta_r)}{\sqrt{-\sum_{r,s}A_rA_s(a_ra_s)^2+\sum_r B_r^2+2\sum_{r\neq s}B_rB_s\cos(\beta_r\beta_s)-2\sum_{r,s}A_rB_s(a_r\beta_s)}}.$$

99. Shew that the normal at a point of a Cartesian oval passes through the symmedian point of the triangle whose vertices are the point and the foci.

100. Find the area of a segment of the curve whose curvature varies as the cube of the sine of the gradient of the tangent.

We have
$$\rho\xi=a\operatorname{cosec}^3(\xi a),$$
$$\therefore\ \rho\xi\,(p\xi a)^2\sin(\xi a)=a\frac{(p\xi a)^2}{\sin^2(\xi a)}.$$

Integrating
$$a\int\frac{(p\xi a)^2}{\sin^2(\xi a)}\,d\xi=\int(xa)^2\sin(\tau xa)\,dx,\quad x=p\xi,$$
$$\therefore\ -2\operatorname{seg}p\xi-(p\xi a)^2\cot(\xi a)=-\frac{1}{3a}(xa)^3=-\frac{1}{3a}(p\xi a)^3.$$
$$\therefore\ \operatorname{seg}p\xi=\frac{1}{6a}(p\xi a)^3-\frac{1}{2}(p\xi a)^2\cot(\xi a).$$

101. Shew that
$$\int(xa)\,dx=\operatorname{lin}x\,(xa)+A\sin(\tau xa)+B\cos(\tau xa),$$
where A, B are intrinsic functions.

102. Shew that
$$\int(xa)^2\,dx=\operatorname{lin}x\,(xa)^2+A\,(\tau xa)+B\,(\nu xa)+C,$$
where A, B, C are intrinsic functions.

103. Shew that
$$\begin{aligned}\int(xa)(x\beta)\,dx=&\operatorname{lin}x\,(xa)(x\beta)+(xa)\{A\sin(\tau x\beta)+B\cos(\tau x\beta)\}\\&+(x\beta)\{C\sin(\tau xa)+D\cos(\tau xa)\}\\&+F\sin(\tau xa)\sin(\tau x\beta)+F\sin(\tau xa)\cos(\tau x\beta)\\&+G\cos(\tau xa)\sin(\tau x\beta)+H\cos(\tau xa)\cos(\tau x\beta),\end{aligned}$$
where A ... H are intrinsic functions.

104. Indicate the general form of the integral of a polynomial function of
$$(xa)^2,\ (xb)^2\ \ldots\ (xa),\ (x\beta)\ \ldots.$$

105. Indicate the general form of the nth differential of a polynomial function of

(i) $(xa)^2,\ (xb)^2\ldots(xa),\ (x\beta)\ldots,$

(ii) $(\xi a),\ (\xi b)\ldots\sin(\xi a),\ \sin(\xi\beta)\ldots.$

106. Shew how to find the family of rhumb lines of the family of curves

(i) $f\{|(xa)|,\ |(xb)|,\ldots(xa),\ (x\beta)\ldots\}=$ variable parameter,

(ii) $f\{(\xi a),\ (\xi b),\ \ldots(\xi a),\ (\xi\beta),\ \ldots\}=$ variable parameter.

ADDITION TO CHAPTER I

To reduce $\cos(\overline{bc}\,\overline{ad})$.

We have

$$|(bc)(ad)|\cos(\overline{bc}\,\overline{ad}) = |(ad)|\,(bc\,\overline{ad}_{\frac{\pi}{2}})$$

$$= |(ad)|\,\{(b\,\overline{ad}_{\frac{\pi}{2}}) - (c\,\overline{ad}_{\frac{\pi}{2}})\}$$

$$= |(ad)|\,\{(ba\,\overline{ad}_{\frac{\pi}{2}}) - (ca\,\overline{ad}_{\frac{\pi}{2}})\}$$

$$= |(ba)(ad)|\cos(\overline{ba}\,\overline{ad}) - |(ca)(ad)|\cos(\overline{ca}\,\overline{ad})$$

Hence $\quad 2\,|(bc)(ad)|\cos(\overline{bc}\,\overline{ad}) = (ac)^2 + (bd)^2 - (ab)^2 - (cd)^2.$

To reduce $\sin(\overline{bc}\,\overline{ad})$.

$$|(bc)(ad)|\sin(\overline{bc}\,\overline{ad}) = (bad) - (cad) = (dbc) - (abc).$$

To reduce $(\overline{\alpha\beta}\,\overline{\gamma\delta})^2$.

$$\sin^2(\alpha\beta)\sin^2(\gamma\delta)(\overline{\alpha\beta}\,\overline{\gamma\delta})^2$$

$$= (\alpha\gamma\delta)^2 + (\beta\gamma\delta)^2 - 2(\alpha\gamma\delta)(\beta\gamma\delta)\cos(\alpha\beta)$$

$$= (\gamma\alpha\beta)^2 + (\delta\alpha\beta)^2 - 2(\gamma\alpha\beta)(\delta\alpha\beta)\cos(\gamma\delta).$$

We shall make another classification of measures.

Measures belong to one of the following three classes :

(i) measures of two points,

(ii) measures of a point and a line,

(iii) measures of two lines.

We shall refer to them as the first, second and third classes respectively.

We have seen that the square of any measure of the first class formed from four elements, any measure of the second class formed from four elements, the sine and cosine of any measure of the third class formed from four elements is reducible to the quotient of two polynomials in the moduli of measures of two points, in the measures of a point and a line, in measures of three points, in measures of three lines, in sines and cosines of measures of two lines, in cosines of measures of two points and a line.

Hence the same is true for measures of five elements: and so on.

We have then that the square of any measure of the first class, any measure of the second class, the sine and cosine of any measure of the third class is reducible to the quotient of two polynomials in

(1) moduli of measures of two points, Ex. $|(ab)|$,

(2) measures of a point and a line, Ex. $(a\beta)$,

(3) sines of measures of two lines, Ex. $\sin(a\beta)$,

(4) cosines of measures of two lines, Ex. $\cos(a\beta)$,

(5) measures of three points, Ex. (abc),

(6) measures of three lines, Ex. $(a\beta\gamma)$,

(7) cosines of measures of two points and a line, Ex. $\cos(\overline{ab}\gamma)$*.

It is often advisable so to reduce any measure, and afterwards reduce cases (5), (7) in surd form, and (6) by the use of a point.

An alternative manner of reducing cases (5), (7) without radicals by multiplying by the sine of two simple lines is given in the Appendix.

* When γ is a direction, both $\sin(\overline{ab}\gamma)$ and $\cos(\overline{ab}\gamma)$ may be taken as irreducible.

APPENDIX

Reduction of Products of Measures

We give four examples of reductions of *products* of measures which possess the property of being reducible without radicals, notwithstanding that the reduction of one or more of the component measures contains a radical. This is due to the eliminants existing between the elements.

The examples are

(1) $(abc)\sin(\alpha\beta)$,

(2) $(abc)\cos(\overline{xy}\,\zeta)\,|(xy)|$,

(3) $|(xy)|\cos(\overline{xy}\,\zeta)\sin(\lambda\mu)$,

(4) $(abc)(\alpha\beta\gamma)$.

(1) Though the reduction of (abc) contains the radical, the product $(abc)\sin(\alpha\beta)$ is expressible without radicals.

The reduction may be effected as follows:

We have

$$(\alpha\beta\overline{bc}) = (a\alpha)\sin(\beta\overline{bc}) + (a\alpha)\sin(\overline{bc}\,\alpha) + (a\overline{bc})\sin(\alpha\beta),$$

$$\therefore\ (abc)\sin(\alpha\beta) = (\alpha\beta bc) + (a\alpha)(bc\beta) - (\alpha\beta)(bc\alpha)$$

$$= \begin{vmatrix} (a\alpha) & (a\beta) & 1 \\ (b\alpha) & (b\beta) & 1 \\ (c\alpha) & (c\beta) & 1 \end{vmatrix}.$$

(2) Here the reductions of both the component measures contain radicals. The product may be reduced as follows:

$$(abc)\cos(\overline{xy}\,\zeta)\,|(xy)|$$

$$= (abc)\sin(\zeta\overline{xy}_{\frac{\pi}{2}})\,|(xy)|$$

$$= \begin{vmatrix} (a\zeta) & (a\overline{xy}_{\frac{\pi}{2}}) & 1 \\ (b\zeta) & (b\overline{xy}_{\frac{\pi}{2}}) & 1 \\ (c\zeta) & (c\overline{xy}_{\frac{\pi}{2}}) & 1 \end{vmatrix}\,|(xy)| \text{ by (1)}$$

$$= \Sigma (a\zeta) \, |(bc)(xy)| \cos(\overline{bc}\,\overline{xy})$$

$$= \tfrac{1}{2}\Sigma (a\zeta) \{(by)^2 + (cx)^2 - (by)^2 - (cx)^2\}$$

$$= \tfrac{1}{2} \begin{vmatrix} (a\zeta) & (ay)^2 - (ax)^2 & 1 \\ (b\zeta) & (by)^2 - (bx)^2 & 1 \\ (c\zeta) & (cy)^2 - (cx)^2 & 1 \end{vmatrix}.$$

(3) Though the reduction of $\cos(\overline{xy}\,\zeta)$ involves radicals, $\cos(\overline{xy}\,\zeta)\sin(\lambda\mu)$ may be reduced without radicals.

From the four lines λ, μ, $\overline{xy}$, ζ we have

$$\sin(\mu\overline{xy})\cos(\lambda\zeta) + \sin(\overline{xy}\lambda)\cos(\mu\zeta) + \cos(\overline{xy}\,\zeta)\sin(\lambda\mu) = 0,$$

$$\therefore \; |(xy)| \cos(\overline{xy}\,\zeta)\sin(\lambda\mu) = \{(x\mu) - (y\mu)\}\cos(\lambda\zeta) - \{(x\lambda) - (y\lambda)\}\cos(\mu\zeta).$$

(4) The product $(abc)(\alpha\beta\gamma)$ may be reduced without radicals as follows:

$$(abc)(\alpha\beta\gamma) = \sum_{\alpha,\beta,\gamma} \sin(\alpha\beta)(\overline{\alpha\beta}\gamma)(abc)$$

$$= \sum_{\alpha,\beta,\gamma} \sin(\alpha\beta) \sum_{a,b,c} -(\overline{\alpha\beta}bc)(a\gamma)$$

$$= \sum_{\substack{a,b,c\\ \alpha,\beta,\gamma}} (a\gamma)(\alpha\beta bc)$$

$$= \sum_{\substack{a,b,c\\ \alpha,\beta,\gamma}} (a\gamma)\{(\alpha b)(\beta c) - (\alpha c)(\beta b)\}.$$

Hence
$$(abc)(\alpha\beta\gamma) = \begin{vmatrix} (a\alpha) & (a\beta) & (a\gamma) \\ (b\alpha) & (b\beta) & (b\gamma) \\ (c\alpha) & (c\beta) & (c\gamma) \end{vmatrix}.$$

Referring to the result on p. 109 and using (1) and (3), it is easy to see that the square of any measure of the first class, any measure of the second class, the sine and cosine of any measure of the third class is reducible to the quotient of two polynomials in

(1) the moduli of measures of two points,

(2) measures of a point and a line,

(3) sines of measures of two lines,

(4) cosines of measures of two lines.

Examples:

1. Shew that $\sin(\overline{bc}\,\overline{ad})$ is reducible rationally by multiplying by $\sin(\lambda\mu)$.

2. Reduce $(ab\gamma d)^2$ by means of the reduction of $(a\beta c)^2$.

We have
$$
\begin{aligned}
(ab\gamma d)^2 &= (ab)^2\,(\overline{ab}\,\gamma d)^2 \\
&= (ab)^2\,\{(\overline{ab}\,d)^2 + (\gamma d)^2 - 2\,(\gamma d)\,(\overline{ab}\,d)\cos(\overline{ab}\,\gamma)\} \\
&= (abd)^2 + (ab)^2(\gamma d)^2 - 2\,(\gamma d)\,|(ab)|\,(abd)\cos(\overline{ab}\,\gamma) \\
&= (abd)^2 + (ab)^2\,(\gamma d)^2 \\
&\qquad - (\gamma d)\begin{vmatrix} (a\gamma) & (ab)^2 & 1 \\ (b\gamma) & -(ab)^2 & 1 \\ (d\gamma) & (db)^2 - (da)^2 & 1 \end{vmatrix} \text{ from (2).}
\end{aligned}
$$

By using the eliminant of three points and a line we get our former reduction.

TABLE OF FORMULAE

FINITE GEOMETRY

CHAPTER I.

$(ba) = -(ab)$ ....(1).

$(\beta a) = (a\beta)$ ....(2).

$(\beta a) = -(a\beta)$ ....(3).

$(a\bar{\beta}) = -(a\beta)$ ....(4).

$(\bar{a}\beta) = (a\beta) + \pi$ ....(5).

$\bar{b}a = \overline{\bar{a}b}$ ....(6).

$\overline{\beta a} = \bar{a}\bar{\beta}$ ....(7).

$a\overline{\beta} = a\bar{\beta}$ ....(8).

$|(a\beta)| = |(\overline{a_{\beta_{\frac{\pi}{2}}}\beta a})|$ ....(9).

$(a_\theta\beta) = (a\beta) - \theta$ ....(10).

$\sin(\beta a) = -\sin(a\beta)$ ....(11).

$\sin(\bar{a}\beta) = -\sin(a\beta)$ ....(12).

$\cos(a\beta) = \sin(a\beta_{\frac{\pi}{2}})$ ....(13).

$\cos(\beta a) = \cos(a\beta)$ ....(14).

$\cos(\bar{a}\beta) = -\cos(a\beta)$ ....(15).

$\sin\frac{\pi}{2} = 1$ ....(16).

$\overline{ab\gamma} = a$ if $(a\gamma) = 0$ ....(17).

$\overline{\overline{a\beta}c} = a$ or $\bar{a}$ if $(ac) = 0$ ....(18).

$(abc) \equiv |(ab)|\,(\bar{a}\bar{b}c)$ ....(19).

$4(a\dot{b}c)^2 = 2(ca)^2(ab)^2 + 2(ab)^2(bc)^2 + 2(bc)^2(ca)^2 - (bc)^4 - (ca)^4 - (ab)^4$...(20).

$(abc) = (bca) = (cab) = -(acb) = -(cba) = -(acb)$ ....(21).

$(ab\gamma) \equiv |(ab)|\sin(\overline{ab}\gamma)$ ....(22).

$(ab\gamma) = (a\gamma) - (b\gamma)$ ....(23).

$$(a\beta c)\equiv \sin(a\beta)\,|(\overline{a\beta}c)| \quad \text{.........(24).}$$

$$(a\beta c)^2=(\overline{a\beta}c)^2\sin^2(a\beta)=(ac)^2+(\beta c)^2-2(ac)(\beta c)\cos(a\beta) \quad \text{......(25).}$$

$$(a\beta\gamma)\equiv\sin(a\beta)\,(\overline{a\beta}\gamma) \quad \text{......(26).}$$

$$(a\beta\gamma)=\sin(\beta\gamma)\,(da)+\sin(\gamma a)\,(d\beta)+\sin(a\beta)\,(d\gamma) \quad \text{......(27).}$$

$$(a\beta\gamma)=(\beta\gamma a)=(\gamma a\beta)=-(\beta a\gamma)=-(\gamma\beta a)=-(a\gamma\beta) \quad \text{......(28).}$$

$$2\,|(ab)\,(cd)|\cos(\overline{ab}\,\overline{cd})=(ad)^2+(bc)^2-(ac)^2-(bd)^2 \quad \text{......(29).}$$

$$|(ab)\,(cd)|\sin(\overline{ab}\,\overline{cd})=(acd)-(bcd)=(dab)-(cab) \quad \text{......(30).}$$

$$(ab\gamma d)\equiv|(ab)|\sin(\overline{ab}\gamma)\,|(\overline{\overline{ab}\gamma}d)| \quad \text{......(31).}$$

$$(ab\gamma d)^2=(bd)^2(a\gamma)^2+(ad)^2(b\gamma)^2+(a\gamma)(b\gamma)\{(ab)^2-(ad)^2-(bd)^2\} \quad \text{...(32).}$$

$$(ab\gamma\delta)\equiv|(ab)|\sin(\gamma\delta)\,(\overline{ab}\,\overline{\gamma\delta}) \quad \text{......(33).}$$

$$(ab\gamma\delta)=(a\gamma)(b\delta)-(a\delta)(b\gamma) \quad \text{......(34).}$$

$$(a\beta cd)\equiv(a\beta c)\,(\overline{\overline{a\beta}c}d) \quad \text{......(35).}$$

$$(a\beta cd)=(cda\beta) \quad \text{......(36).}$$

$$(a\beta c)\sin(\overline{\overline{a\beta}c}\,\delta)=(ac)\sin(\beta\delta)-(\beta c)\sin(a\delta) \quad \text{......(37).}$$

$$(a\beta c)\cos(\overline{\overline{a\beta}c}\,\delta)=(ac)\cos(\beta\delta)-(\beta c)\cos(a\delta) \quad \text{......(38).}$$

$$\sin^2(a\beta)\sin^2(\gamma\delta)(\overline{a\beta}\,\overline{\gamma\delta})^2=(a\beta\gamma)^2+(a\beta\delta)^2-2(a\beta\gamma)(a\beta\delta)\cos(\gamma\delta)$$
$$=(\gamma\delta a)^2+(\gamma\delta\beta)^2-2(\gamma\delta a)(\gamma\delta\beta)\cos(a\beta) \quad \text{...(39).}$$

If a, b, c be three points on a line, then

$$(bc)+(ca)+(ab)=0 \quad \text{......(40).}$$

If a, b, c, d be four points on a line, and $ab=cd$, then $(ab)/(cd)$ is positive ... (41).

$$(\beta\gamma)+(\gamma a)+(a\beta)=2m\pi,\ m \text{ an integer} \quad \text{......(42).}$$

$$\left.\begin{aligned}\sin(\theta+\phi)&=\sin\theta\cos\phi+\sin\phi\cos\theta\\ \cos(\theta+\phi)&=\cos\theta\cos\phi-\sin\theta\sin\phi\end{aligned}\right\} \quad \text{......(43).}$$

$$\begin{vmatrix}0 & 1 & 1 & 1 & 1\\ 1 & 0 & (ab)^2 & (ac)^2 & (ad)^2\\ 1 & (ba)^2 & 0 & (bc)^2 & (bd)^2\\ 1 & (ca)^2 & (cb)^2 & 0 & (cd)^2\\ 1 & (da)^2 & (db)^2 & (dc)^2 & 0\end{vmatrix}=0 \quad \text{......(44).}$$

$$(a\delta)^2(bc)^2+(b\delta)^2(ca)^2+(c\delta)^2(ab)^2$$
$$-2\,|(ca)(ab)|\cos(\overline{ca}\,\overline{ab})\,(b\delta)(c\delta)-2\,|(ab)(bc)|\cos(\overline{ab}\,\overline{bc})\,(c\delta)(a\delta)$$
$$-2\,|(bc)(ca)|\cos(\overline{bc}\,\overline{ca})\,(a\delta)(b\delta)=(abc)^2 \quad \text{......(45).}$$

$$(ab)^2\sin^2(\gamma\delta)=\{(b\gamma)-(a\gamma)\}^2+\{(b\delta)-(a\delta)\}^2$$
$$-2\{(b\gamma)-(a\gamma)\}\{(b\delta)-(a\delta)\}\cos(\gamma\delta) \quad \text{......(46).}$$

CHAPTER II.

$$(a_{\acute\rho\acute\sigma\ldots\acute\omega}b)^2=(ab)^2-2\,|(ab)|\,\Sigma\,\hat\rho\cos(\overline{ab}\rho)+\Sigma\,\hat\rho^2+2\,\Sigma\,\hat\rho\,\hat\sigma\cos(\rho\sigma) \quad \text{...(47).}$$

$$(a_{\acute\rho\acute\sigma\ldots\acute\omega}\beta)=(a\beta)-\Sigma\,\hat\rho\sin(\rho\beta) \quad \text{......(48).}$$

$$(a_{\acute\rho\acute\sigma\ldots\acute\phi_\omega}b)=-(ab\omega)+\Sigma\,\hat\rho\sin(\rho\omega) \quad \text{......(49).}$$

$$(a_{\acute\rho\acute\sigma\ldots\acute\phi_\omega}\beta)=(\omega\beta) \quad \text{......(50).}$$

CHAPTER III.

$$\sin(\Sigma a_r(xa_r)+a=0\ \beta)=\frac{\Sigma a_r \sin(a_r\beta)}{m\,|\sqrt{\Sigma a_r^2+2\Sigma a_r a_s \cos(a_r a_s)}\,|} \quad\text{.........(51).}$$

$$\cos(\Sigma a_r(xa_r)+a=0\ \beta)=\frac{\Sigma a_r \cos(a_r\beta)}{m\,|\sqrt{\Sigma a_r^2+2\Sigma a_r a_s \cos(a_r a_s)}\,|} \quad\text{.........(52).}$$

$$(\Sigma a_r(xa_r)+a=0\ b)=\frac{\Sigma a_r(ba_r)+a}{m\ \sqrt{\Sigma a_r^2+2\Sigma a_r a_s \cos(a_r a_s)}\,|} \quad\text{.........(53).}$$

We suppose $\Sigma a_r(xa_r)+a=0$ to have a specified sense, and m is $+1$ or -1.

$$(\Sigma A_r(\xi a_r)+\Sigma B_r \cos(\xi\beta_r)=0\ c)^2(\Sigma A_r)^2$$
$$=\Sigma A_r^2(a_r c)^2+2\sum_{r\neq s} A_r A_s|(a_r c)(a_s c)|\cos(\overline{a_r c}\,\overline{a_s c})$$
$$+2\Sigma A_r B_s(ca_r\beta_s)+\Sigma B_r^2+2\Sigma B_r B_s\cos(\beta_r\beta_s)\quad\text{............(54).}$$

$$(\Sigma A_r(\xi a_r)+\Sigma B_r\cos(\xi\beta_r)=0\ \gamma)(\Sigma A_r)=\Sigma A_r(\gamma a_r)+\Sigma B_r\cos(\gamma\beta_r).\quad\text{.(55).}$$

When $\Sigma A_r=0$,

$$\tan(\Sigma A_r(\xi a_r)+\Sigma B_r\cos(\xi\beta_r)=0\ \gamma)$$
$$=\frac{\Sigma A_r(a_r\gamma)+\Sigma B_r\cos(\beta_r\gamma)}{\Sigma A_r(a_r\gamma_{\frac{\pi}{2}})-\Sigma B_r\sin(\beta_r\gamma)}\quad\text{......................(56).}$$

DIFFERENTIAL GEOMETRY

CHAPTER IV.

$$d\,|(xy)|=-\cos(\tau xxy)\,dx-\cos(\tau y\,\overline{yx})\,dy.\ \text{.. ....(57)}$$

$$=\frac{(\nu xy)}{|(xy)|}dx+\frac{(\nu yx)}{|(xy)|}dy\text{...... ....(58).}$$

$$d(x\eta)=-\sin(\tau x\eta)\,dx+(x\nu\eta)\,d\eta\ \text{. . . . .(59).}$$

$$d(\xi\eta)=d\eta-d\xi\ \text{. .... (60)}$$

CHAPTER V.

$$(xy)^2\,dxy=(y\tau x)\,dx+(x\tau y)\,dy\ \text{......... ....(61).}$$

$$\sin^4(\xi\eta)(d\overline{\xi\eta})^2=(p\xi\eta)^2(d\xi)^2+(p\eta\xi)^2(d\eta)^2+2(p\xi\eta)(p\eta\xi)\cos(\xi\eta)\,d\xi d\eta\text{...(62)}$$

From Chapter VIII

$$(xy)^2(pxya)\,d\overline{xy}=(\tau xy)(ya)\,dx+(\tau yx)(xa)\,dy\ \text{..... ...(63).}$$

$$\sin^2(\xi\eta)(\tau\overline{\xi\eta}a)\,d\overline{\xi\eta}=(p\xi\eta)(\eta a)\,d\xi+(p\eta\xi)(\xi a)\,d\eta\text{.... ....(64).}$$

CHAPTER VI.

$$(dx_{\rho\sigma\ \omega})^2=(dx)^2+\Sigma(d\hat\rho)^2+\Sigma\hat\rho^2 d\rho^2$$
$$+2\,dx\,\Sigma\,d\hat\rho\cos(\tau x\rho)-2dx\,\Sigma\,\hat\rho\,d\rho\sin(\tau x\rho)$$
$$+\Sigma\,d\hat\rho^2+\Sigma\hat\rho^2 d\rho^2-2\Sigma(d\hat\rho\,\hat\sigma\,d\sigma-d\hat\sigma\,\hat\rho\,d\rho)\sin(\rho\sigma)$$
$$+2\Sigma(d\hat\rho\,d\hat\sigma+\hat\rho\hat\sigma\,d\rho\,d\sigma)\cos(\rho\sigma)\text{............ .. ..(65).}$$

$$dx_{\rho\sigma\ \phi\omega}=d\omega\ \text{.... ....(66).}$$

CHAPTER VII.

$$d\{\Sigma a_r(xa_r)+a=0\}=\{\Sigma(a_r da_s - a_s da_r)\sin(a_r a_s)+\Sigma a_r^2 da_r$$
$$+\sum_{r\neq s} a_r a_s \cos(a_r a_s)(da_r+da_s)\}\div\{\Sigma a_r^2+2\,\Sigma\, a_r a_s\cos(a_r a_s)\} \quad\ldots\ldots(67).$$

Hence $(da)^2 . (\sum_r A_r)^4$

$$=\sum_r (dA_r)^2\{\sum_h A_h^2(a_r a_h)^2+2\sum_{h\neq k} A_h A_k|(a_r a_h)(a_r a_k)|\cos(\overline{a_r a_h}\,\overline{a_r a_k})$$
$$+2\sum_{h,k} A_h B_k(a_h a_r \beta_k)+\sum_h B_h^2+2\sum_{h\neq k} B_h B_k\cos(\beta_h\beta_k)\}$$
$$+\sum_{r\neq s} dA_r dA_s\{-(a_r a_s)^2(\sum_h A_h)^2+\sum_h A_h^2(a_r a_h)^2$$
$$+\sum_{h\neq k} A_h A_k|(a_r a_h)(a_r a_k)|\cos(\overline{a_r a_h}\,\overline{a_r a_k})$$
$$+2\sum_{h,k} A_h B_k(a_h a_r\beta_k)+2\sum_h B_h^2+4\sum_{h\neq k} B_h B_k\cos(\beta_h\beta_k)$$
$$+\sum_h A_h^2(a_r a_h)^2+\sum_{h\neq k} A_h A_k|(a_s a_h)(a_s a_k)|\cos(\overline{a_s a_h}\,\overline{a_s a_k})$$
$$+2\sum_{h,k} A_h B_k(a_h a_s\beta_k)\}$$
$$+2\sum_h A_h . \sum_{r,s} dA_r A_s da_s\{\sum_h A_h(a_h a_r \nu a_s)-\sum_h B_h\sin(\beta_h \nu a_s)\}$$
$$-2\sum_h A_h . \sum_{r,s} dA_r dB_s\{\sum_h A_h(a_h a_r\beta_s)+\sum_h B_h\cos(\beta_h\beta_s)\}$$
$$-2\sum_h A_h . \sum_{r,s} dA_r B_s d\beta_s\{\sum_h A_h(a_h a_s\nu\beta_s)-\sum_h B_h\sin(\beta_h\beta_s)\}$$
$$+(\sum_h A_h)^2[\sum_r(dB_r)^2+\sum_r B_r^2(d\beta_r)^2+2\sum_{r\neq s} A_r A_s\cos(\tau a_r\tau a_s)\,da_r da_s$$
$$+2\sum_{r\neq s} dB_r dB_s\cos(\beta_r\beta_s)+2\sum_{r\neq s} B_r B_s d\beta_r d\beta_s\cos(\beta_r\beta_s)$$
$$-2\sum_{r,s} A_r da_r dB_s\sin(\tau a_r\beta_s)-\sum_{r,s} A_r da_r B_s d\beta_r\cos(\tau a_r\beta_s)$$
$$-2\sum_{r,s} dB_r B_s d\beta_s\sin(\beta_r\beta_s)] \quad\ldots\ldots(68).$$

CHAPTER VIII.

$$(\nu xa)=\frac{1}{2}\frac{d(xa)^2}{dx} \quad\ldots\ldots(69).$$

$$(\tau xa)=(\nu a)^2\frac{d\overline{xa}}{dx} \quad\ldots\ldots(70).$$

$$\sin(\tau xa)=-\frac{d(xa)}{dx} \quad\ldots\ldots(71).$$

$$\cos(\nu xa)=-\frac{d(xa)}{dx} \quad\ldots\ldots(72).$$

$$(p\xi a)^2=(\xi a)^2+\left[\frac{d(\xi a)}{d\xi}\right]^2 \quad\ldots\ldots(73).$$

$$\left.\begin{aligned}(p\xi a)&=\sin(a\xi)\left[\frac{d(\xi a)}{d\xi}\right]_{a=\bar a\xi}\\ (p\xi a)&=-\frac{\sin(a\beta)}{\sin^2(\xi a)}\frac{d}{d\xi}(\bar\xi a\beta)\end{aligned}\right\} \quad\ldots\ldots(74).$$

$$(\nu\xi a)=\frac{d(\xi a)}{d\xi} \quad\ldots\ldots(75).$$

$$(\nu\xi a)=(\xi a)-\frac{\pi}{2} \quad\ldots\ldots(76).$$

CHAPTER IX.

If the displacement of x be

$$\tau_1 x,\ \delta_1 x;\ \tau_2 x,\ \delta_2 x;\ \ldots\ \tau_n x,\ \delta_n x,$$

then

$$(dx)^2 = \Sigma\, d_r x^2 + 2\Sigma\, d_r x d_s x \cos(\tau_r x \tau_s x) \quad \text{(77)}.$$

$$d\tau x\,(dx)^2 = \Sigma\, d_r x^2 d\tau_r x + \Sigma\, d_r x\, d_s x\,(d\tau_r x + d\tau_s x)\cos(\tau_r x \tau_s x)$$
$$+ \Sigma\,(d_r x d_s^2 x - d_s x d_r x^2)\sin(\tau_r x \tau_s x) \quad \text{(78)}.$$

$$(\nu x a)\, dx = \Sigma\, d_r x\,(x_{\nu_r x} a) \quad \text{(79)}.$$

$$(\tau x a)\, dx = \Sigma\, d_r x\,(x_{\tau_r x} a) \quad \text{(80)}.$$

PLANE CURVES

CHAPTER X.

$$p\tau x = x \quad \text{(81)}.$$

$$\nu\tau x = \nu x \quad \text{(82)}.$$

$$(\nu^2 x a) = \frac{1}{\rho x} - (\tau x a) \quad \text{(83)}.$$

$$(p\nu x a) = (x a) + \frac{\cos(\tau x a)}{\rho x} \quad \text{(84)}.$$

$$\tau p\xi = \xi \quad \text{(85)}.$$

$$\nu p\xi = \nu\xi \quad \text{(86)}.$$

$$(\nu^2 \xi a) = \rho\xi - (\xi a) \quad \text{(87)}.$$

$$(p\nu\xi a) = (p\xi a) + \cos(\xi a)\,\rho\xi \quad \text{(88)}$$

CHAPTER XI.

$$\frac{1}{2}\frac{d}{dx}(xa)^2 = (\nu x a) \quad \text{(89)}.$$

$$\frac{1}{2}\left(\frac{d}{dx}\right)^2 (xa)^2 = 1 - (\tau x a)\,\rho x \quad \text{(90)}.$$

$$\frac{1}{2}\left(\frac{d}{dx}\right)^3 (xa)^2 = -(\rho x)^2(\nu x a) - \rho' x\,(\tau x a) \quad \text{(91)}.$$

$$\frac{1}{2}\left(\frac{d}{dx}\right)^4 (xa)^2 = -(\rho x)^2 + \{(\rho x)^3 - \rho'' x\}(\tau x a) - 3\rho x \rho' x\,(\nu x a) \quad \text{(92)}.$$

$$\frac{1}{2}\left(\frac{d}{dx}\right)^5 (xa)^2 = -5\rho x \rho' x + \{(\rho x)^4 - 4\rho x \rho'' x - 3(\rho' x)^2\}(\nu x a)$$
$$+ \{6(\rho x)^2 \rho' x - \rho''' x\}(\tau x a) \quad \text{(93)}.$$

And generally, $\frac{1}{2}\left(\frac{d}{dx}\right)^n (xa)^2 = A_n(\tau x a) + B_n(\nu x a) + C_n$,

where A, B, C are polynomials of ρx, $\rho' x$, $\rho'' x$ ..., and

$$A_{n+1} = A_n' - B_n \rho x,$$
$$B_{n+1} = B_n' + A_n \rho x,$$
$$C_{n+1} = C_n' + B_n.$$

$$\frac{d}{dx}|(xa)| = \frac{(\nu x a)}{|(xa)|} \quad \text{(94)}.$$

$$\left(\frac{d}{dx}\right)^2 |(xa)| = \frac{1-\rho x\,(\tau xa)}{|(xa)|} - \frac{(\nu xa)^2}{|(xa)|^3} \quad \ldots\ldots(95).$$

$$\left(\frac{d}{dx}\right)^3 |(xa)| = -\frac{\rho' x\,(\tau xa)}{|(xa)|} - \frac{(\rho x)^2\,(\nu xa)}{|(xa)|} - \frac{3\,(\nu xa)}{|(xa)|^3}$$
$$+\frac{3\rho x\,(\tau xa)\,(\nu xa)}{|(xa)|^3} + \frac{3\,(\nu xa)^3}{|(xa)|^5} \quad \ldots\ldots(96).$$

$$\left(\frac{d}{dx}\right)^4 |(xa)| = \frac{(\rho x)^2}{|(xa)|} - \frac{3}{|(xa)|^3} + (\tau xa)\left\{-\frac{\rho'' x}{|(xa)|} + \frac{(\rho x)^3}{|(xa)|} - \frac{6\rho x}{|(xa)|^3}\right\}$$
$$-(\nu xa)\frac{3\rho x\rho' x}{|(xa)|} + (\tau xa)\,(\nu xa)\frac{4\rho' x}{|(xa)|^3}$$
$$-(\tau xa)^2\frac{3\,(\rho x)^2}{|(xa)|^3} + (\nu xa)^2\left\{\frac{4\,(\rho x)^2}{|(xa)|^3} - \frac{18}{|(xa)|^5}\right\}$$
$$-(\tau xa)\,(\nu xa)^2\frac{18\rho x}{|(xa)|^5} - \frac{15\,(\nu xa)^4}{|(xa)|^7} \quad \ldots\ldots(97).$$

$$\frac{d\,(xa)}{dx} = -\sin(\tau xa) \quad \ldots\ldots(98).$$

$$\left(\frac{d}{dx}\right)^2 (xa) = \rho x \cos(\tau xa) \quad \ldots\ldots(99).$$

$$\left(\frac{d}{dx}\right)^3 (xa) = \rho' x \cos(\tau xa) + (\rho x)^2 \sin(\tau xa) \quad \ldots\ldots(100).$$

$$\left(\frac{d}{dx}\right)^4 (xa) = \{\rho'' x - (\rho x)^3\} \cos(\tau xa) + 3\rho x\rho' x \sin(\tau xa) \quad \ldots\ldots(101).$$

$$\left(\frac{d}{dx}\right)^5 (xa) = \{6\,(\rho x)^2\rho' x - \rho''' x\} \cos(\tau xa)$$
$$+\sin(\tau xa)\,\{4\rho x\rho'' x + 3\,(\rho' x)^2 - (\rho x)^4\} \quad (102).$$

And generally, $\left(\frac{d}{dx}\right)^n (xa) = A_n \sin(\tau xa) + B_n \cos(\tau xa)$,

where A, B are polynomials in ρx, $\rho' x$, $\rho'' x$ ... and

$$A_{n+1} = A_n' + B_n\rho x,\quad B_{n+1} = B_n' - A_n\rho x.$$

$$\left(\frac{d}{d\xi}\right)^{2n} (\xi a) = \rho^{(2n-2)}\xi - \rho^{(2n-4)}\xi + \ldots + (-)^{n-1}\rho\xi + (-)^n(\xi a) \quad \ldots(103).$$

$$\left(\frac{d}{d\xi}\right)^{2n-1} (\xi a) = \rho^{(2n-3)}\xi - \rho^{(2n-5)}\xi + \ldots + (-)^n\rho'\xi + (-)^{n-1}(\nu\xi a). \quad \ldots(104).$$

APPENDIX.

$$(abc)\sin(a\beta) = \begin{vmatrix} (aa) & (a\beta) & 1 \\ (ba) & (b\beta) & 1 \\ (ca) & (c\beta) & 1 \end{vmatrix} \quad \ldots\ldots(105)$$

$$|(ab)|\cos(a\bar{b}\gamma)\sin(\lambda\mu) = \{(a\mu)-(b\mu)\}\cos(\lambda\gamma) - \{(a\lambda)-(b\lambda)\}\cos(\mu\gamma) \ldots(106).$$

$$2\,(abc)\cos(x\bar{y}\zeta)\,|(xy)| = \begin{vmatrix} (a\zeta) & (ay)^2-(ax)^2 & 1 \\ (b\zeta) & (by)^2-(bx)^2 & 1 \\ (c\zeta) & (cy)^2-(cx)^2 & 1 \end{vmatrix} \quad \ldots\ldots(107).$$

$$(abc)\,(a\beta\gamma) = \begin{vmatrix} (aa) & (a\beta) & (a\gamma) \\ (ba) & (b\beta) & (b\gamma) \\ (ca) & (c\beta) & (c\gamma) \end{vmatrix} \quad \ldots\ldots(108).$$

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کتب خانہ

جامعہ عثمانیہ

۱- اراکین مجلس اعلیٰ، مجلس رفقا، مجلس انتظامی، مجلس تعلیمات، نظما پانچ کتابیں ایک ماہ تک اپنے پاس رکھ سکیں گے۔

۲- اساتذہ جامعہ عثمانیہ و کلیات ملحقہ و رجسٹرار و معتمدان جامعہ دس کتابیں ایک ماہ تک اپنے پاس رکھ سکیں گے۔

اور اراکین دارالترجمہ دس کتابیں لے سکتے ہیں۔ اور انٹرمیڈیٹ

۳- بی اے، ایم اے، ال ال بی و تحتانی جماعتوں کے طلبہ تین کتابیں ایم اے کے طلبہ ... اپنے پاس رکھ سکیں گے۔

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۶- کتابیں گم یا خراب ہو جائیں تو مستعیر پر ذمہ داری عائد ہوگی۔

۷- کتابوں پر کسی قسم کا نشان سیاہی یا پنسل سے نہ لگایا جائے۔

۸- قلمی نسخے اور حوالے کی کتابیں جاری نہ کی جا سکینگی۔